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Besov regularity and Pointwise regularity

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1 Introduction

There are many ways to characterize Besov spaces. Among them in the discrete version are regular wavelet expansion, Littlewood-Paley decomposition, polynomial approximation, spline approximation, mean oscillation, and difference operator (See [Meyer], [Triebel] and [Wojtaszczyk]).

We apply these characterizations to study these pointwise versions. In particular we consider a characterization in a framework of multiresolution approximation and give conditions of finitely many functions which generate Besov spaces in a view of multiresolution approximation scheme. This result is a generalization of characterizations of Besov spaces given by regular wavelet functions and by spline functions (See [Sickel] and [Wojtaszczyk]). Moreover we investigate to give descriptions of scaling exponents for Besov regularity and pointwise regularity.

We give scaling exponents for Besov regularity of some spline series and estimates of a pointwise Hölder exponent of self-similar series. We apply this result to compute pointwise Hölder exponents of several oscillatory functions (ref. [Jaffard 1]).

The plan of sections in our paper is as follows:

In the second section we give characterizations of Besov spaces and those pointwise versions.

In the third section we consider a multiresolution analysis $\{V_l\}$ generated by finitely many functions and give properties of Besov space norms defined by approximation errors associated with $\{V_l\}$.

In the fourth section we give some conditions of finitely many functions which characterize a Besov space by multiresolution approximation. This result gives a generalization of characterizations of Besov spaces given by regular wavelet functions and by spline functions, and we also give characterizations of the pointwise Hölder space by multiresolution approximation.

In the fifth section we give descriptions of scaling exponents of Besov regularity and poitwise regularity by characterizations of Besov spaces.

Moreover, we give scaling exponents for Besov regularity of some spline series.

In sixth section we give estimates of a pointwise Hölder exponent for a self-similar series and apply to compute a pointwise Hölder exponent of several oscillatory functions(e. g. Takagi functions, Weierstrass function and Lèvy's functions).

We use C to denote a positive constant different in each occasion. But it will depend on the parameter appearing in each problem. The same notations C are not necessarily the same on any two occurrences.

2 Besov spaces and pointwise regularity

Let Q_0 be a unit cube with sides parallel to the axes of coordinates, centered at the origin in $\mathbb{R}^n$. We define for a function f on $\mathbb{R}^n$,

$$\operatorname{osc}_p^k f(x,l) = \inf_{P \in \mathbb{P}^k} \Big(\frac{1}{|Q_l(x)|} \int_{Q_l(x)} |f(y) - P(y)|^p dy\Big)^{1/p} \tag{1}$$

where $Q_l(x) = 2^{-l}Q_0 + x$ and $|Q_l(x)|$ is the volume element of $Q_l(x)$, and $\mathbb{P}^k$ is the linear space of all polynomials of degree no greater than k on $\mathbb{R}^n$.

Definition.

Given $s > 0$, k a nonnegative integer with $k+1 > s$ and $1 \le p, q \le \infty$.

A function f is said to belong to the Besov space $B^s_{pq}(\mathbb{R}^n)$ if

$$||f||_{B^s_{pq}(\mathbb{R}^n)} = ||f||_p + \Big(\sum_{l=0}^{\infty} (2^{ls} ||\operatorname{osc}_p^k f(\cdot, l)||_p)^q\Big)^{1/q} < \infty. \tag{2}$$

with the usual modification for $q = \infty$. We note that the above definition is independent of the choice of nonnegative integers k with $k+1 > s$ and osc_p^k in the definition can be replaced by osc_1^k. We can see $W^p_{k+1}(\mathbb{R}^n) \subset B^s_{pq}(\mathbb{R}^n)$ if $s < k+1$. For a domain D of $\mathbb{R}^n$, we define by restriction $B^s_{pq}(D) = B^s_{pq}(\mathbb{R}^n)|_D$.

Let $\triangle_u f$ denote the difference operator $\triangle_u f(x) = f(x+u) - f(x)$.

Theorem A ([Triebel]). *Given $s > 0$, a nonnegative integer k with $k+1 > s$ and $1 \le p, q \le \infty$.*

Then we have equivalent ones of the Besov space norm given in (2),*if one of them exists, with the usual modification for* $q = \infty$*,*

$$||f||_{B^s_{pq}(\mathbb{R}^n)} \sim ||f||_p + (\sum_{l=0}^{\infty}(2^{ls} \sup_{|u|<2^{-l}} ||\triangle_u^{k+1} f||_p)^q)^{1/q}.$$

For $x \in \mathbb{R}^n$, a bounded function $f \in T^s_{pq}(x)$ means that

$$(\sum_{l=0}^{\infty}(2^{ls}\mathrm{osc}^k_p f(x,l))^q)^{1/q} < \infty$$

with the usual modification for $q = \infty$.

We have the embedding theorem : $T^\beta_{p\xi}(x) \subset T^\alpha_{p\eta}(x)$ for $\beta > \alpha > 0$, $1 \le \xi, \eta \le \infty$ and $1 \le p \le \infty$, and $T^\alpha_{p\eta}(x) \subset T^\alpha_{p\xi}(x)$, $T^\alpha_{\xi q}(x) \subset T^\alpha_{\eta q}(x)$ for $\alpha > 0$, $1 \le \eta \le \xi \le \infty$ and $1 \le p, q \le \infty$.

We write $T^s_{\infty\infty}(x) = C^s(x)$.

Theorem 1 ([Saka 1]). *Given* $s > 0$ *, a nonnegative integer* k *with* $k+1 > s$ *and* $1 \le p, q \le \infty$*.*

Then for $x \in \mathbb{R}^n$ *following statements of a bounded function* f *are equivalent, with the usual modification for* $q = \infty$*,*

(i) $f \in T^s_{pq}(x)$,

(ii) $(\sum_{l=0}^{\infty}(2^{ls} \sup_{|u|<2^{-l}}(\frac{1}{|Q_l(x)|}\int_{Q_l(x)} |\triangle_u^{k+1} f(y)|^p dy)^{1/p})^q)^{1/q} < \infty.$

We will define the Littlewood-Paley decomposition. Let φ be a function in the Schwartz class $\mathcal{S}(\mathbb{R}^n)$ with the following properties: supp $\hat{\varphi} \subset \{\xi \in \mathbb{R}^n : |\xi| \le 1\}$ and $\hat{\varphi}(\xi) = 1$ on $\{\xi \in \mathbb{R}^n : |\xi| \le 2^{-1}\}$. Let $\psi(x) = 2^n\varphi(2x) - \varphi(x)$. Let $\varphi_l(x) = 2^{ln}\varphi(2^l x)$, $S_l f = f * \varphi_l$, $\psi_l(x) = 2^{ln}\psi(2^l x)$ and $f_l = f * \psi_l$ for $l = 0, 1, 2, \ldots$. Then for $f \in \mathcal{S}'$ we have Littlewood-Paley decomposition:

$$f = \varphi * f + \sum_{l=0}^{\infty} \psi_l * f \equiv S_0 f + \sum_{l=0}^{\infty} f_l. \tag{3}$$

Theorem B ([Triebel]). *Let* $1 \le p, q \le \infty$ *and* $s > 0$*.*

Then we have equivalence of norms if one of them exists, for Littlewood-Paley decomposition given in (3)*, with the usual modification* $q = \infty$*:*

(i) $||f||_{B^s_{pq}(\mathbb{R}^n)}$,

$\sim$ (ii) $||f||_p + (\sum_{l=0}^{\infty}(2^{ls}||f - S_l f||_p)^q)^{1/q}$,

$\sim$ (iii) $||S_0 f||_p + (\sum_{l=0}^{\infty}(2^{ls}||f_l||_p)^q)^{1/q}$.

The following statement is a pointwise version of Theorem B and can be proved by Theorem 1 using the same way as in [Andersson].

Theorem 2 ([Saka 1]). *Let $s > 0$ and f a bounded function on $\mathbb{R}^n$.*

(a) *For $x \in \mathbb{R}^n$, following statements are equivalent:*

(i) $f \in C^s(x)$,
(ii) $|f(y) - S_l f(y)| \leq C(2^{-l} + |x-y|)^s$ *for all* $l = 0, 1, 2, \cdots$.

(b) *If $f \in C^s(x)$, then it holds*

(iii) $|f_l(y)| \leq C(2^{-l} + |x-y|)^s$ *for all* $l = 0, 1, 2, \cdots$.

Conversely, if it holds for $s > s' > 0$,

(iii)$'$ $|f_l(y)| \leq C2^{-ls}(1 + 2^l|x-y|)^{s'}$ *for all* $l = 0, 1, 2, \cdots$,

then $f \in C^s(x)$.

3 Multiresolution approximation

For $1 \leq p \leq \infty$, let $\mathcal{L}^p = \mathcal{L}^p(\mathbb{R}^n)$ be the linear space of all functions ϕ for which

$$|\phi|_p = (\int_T (\sum_{\nu \in \mathbb{Z}^n} |\phi(x-\nu)|)^p dx)^{1/p} < \infty. \tag{4}$$

with the usual modification for $p = \infty$ and the unit cube $T = [0,1]^n$. Clearly, $\mathcal{L}^p \subset L^p(\mathbb{R}^n)$ and $\mathcal{L}^\infty \subset \mathcal{L}^p \subset \mathcal{L}^q \subset \mathcal{L}^1 = L^1(\mathbb{R}^n)$ for $1 \leq q \leq p \leq \infty$. If $\phi \in L^p(\mathbb{R}^n)$ $(1 \leq p \leq \infty)$ is compactly supported, then $\phi \in \mathcal{L}^p$. Furthermore, we observe that if there are constants $C > 0$ and $\delta > 0$ such that $|\phi(x)| \leq C(1+|x|)^{-n-\delta}$ for all $x \in \mathbb{R}^n$ then $\phi \in \mathcal{L}^\infty$.

A finite subset $\Phi = \{\phi_1, \dots, \phi_N\}$ of $\mathcal{L}^\infty$ is said to have L^p-stable shifts $(1 \le p \le \infty)$, if there are constants $C_1 > 0$ and $C_2 > 0$ such that for any sequences $c_j \in l^p(\mathbb{Z}^n)$ $(j = 1, \dots, N)$,

$$C_1 \sum_{j=1}^{N} ||c_j||_{l^p} \le || \sum_{j=1}^{N} \sum_{\nu \in \mathbb{Z}^n} c_j(\nu)\phi_j(x-\nu)||_p \le C_2 \sum_{j=1}^{N} ||c_j||_{l^p}.$$

Theorem C ([Jia-Micchelli]). *For a finite subset $\Phi = \{\phi_1, \dots, \phi_N\}$ of $\mathcal{L}^\infty$, we have following equivalent conditions:*

(i) Φ *has L^2-stable shifts,*
(ii) Φ *has L^p-stable shifts for $1 \le p \le \infty$,*
(iii) *there is a set of functions $\tilde{\Phi} = \{\tilde{\phi}_1, \dots, \tilde{\phi}_N\}$ in $\mathcal{L}^\infty$, dual to Φ in the sense that*

$$\int \phi_j(x-\mu)\bar{\tilde{\phi}}_k(x-\nu)dx = \delta_{\mu\nu}\delta_{jk}, \quad j,k = 1, \dots, N, \quad \mu, \nu \in \mathbb{Z}^n,$$

where δ is the Kronecker's symbol.

Let $\Pi = \{T + \nu\}_{\nu \in \mathbb{Z}^n}$ with the unit cube $T = [0,1]^n$. For an integer l and a finite subset $\Phi = \{\phi_1, \dots, \phi_N\}$ of $\mathcal{L}^\infty$ with L^2-stable shifts, we define operators $P_l f$ given by

$$P_l f(x) = \sum_{j=1}^{N} \sum_{\nu \in \mathbb{Z}^n} 2^{nl} \langle f, \tilde{\phi}_j(2^l \cdot - \nu)\rangle \phi_j(2^l x - \nu) \qquad (5)$$

where $\langle f, \tilde{\phi}_j(2^l \cdot - \nu)\rangle = \int f(y)\bar{\tilde{\phi}}_j(2^l y - \nu)\, dy$ and $\tilde{\Phi} = \{\tilde{\phi}_1, \dots, \tilde{\phi}_N\}$ is dual to Φ in Theorem C.

Let $V_0^p = \{\sum_{j=1}^{N} \sum_{\nu \in \mathbb{Z}^n} a_j(\nu)\phi_j(x-\nu) : \quad a_j \in l^p(\mathbb{Z}^n)\}$ and let $V_l^p = \{f(2^l x) : f \in V_0^p\}$. Then for $1 \le p \le \infty$, the operator P_l is a bounded projection operator of $L^p(\mathbb{R}^n)$ onto V_l^p $(1 \le p \le \infty)$ in the sense that $P_l f = f$ for any $f \in V_l^p$.

We say $\Phi = \{\phi_1, \dots, \phi_N\}$ of $\mathcal{L}^\infty$ is refinable if there exist sequences $c_{jk} \in l^1(\mathbb{Z}^n)$ $(1 \le j, k \le N)$ such that

$$\phi_j(x) = \sum_{k=1}^{N} \sum_{\nu \in \mathbb{Z}^n} c_{jk}(\nu)\phi_k(2x-\nu), \quad x \in \mathbb{R}^n, \quad j = 1, \dots, N.$$

A following theorem implies that $\{V_l^p\}$ is a multiresolution analysis in $L^p(\mathbb{R}^n)$ for $1 \le p < \infty$.

Theorem D ([Jia-Micchelli] and [Zhao]). *If a finite subset Φ of $\mathcal{L}^\infty$ is refinable and has L^2-stable shifts, then the sequence of sets $\{V_l^p\}$ $(1 \le p \le \infty)$ satisfies following properties:*

(i) $f \in V_0^p \Leftrightarrow f(x-\nu) \in V_0^p$ *for all* $\nu \in \mathbb{Z}^n$,
(ii) $f \in V_l^p \Leftrightarrow f(2x) \in V_{l+1}^p$,
(iii) $\cdots \subset V_l^p \subset V_{l+1}^p \subset \cdots$,
(iv) $\cap_{l\in\mathbb{Z}} V_l^p = \{0\}$ $(1 \le p < \infty)$,
(v) $\cup_{l=0}^\infty V_l^p$ *is dense in* $L^p(\mathbb{R}^n)$ $(1 \le p < \infty)$.

Assume that Φ of $\mathcal{L}^\infty$ satisfies conditions of Theorem D.

Given a function f in $L^p(\mathbb{R}^n)$ $(1 \le p \le \infty)$, $\sigma_l^p(f)$ denotes the error of L^p-approximation from V_l^p in $L^p(\mathbb{R}^n)$:

$$\sigma_l^p(f) = \inf\{||f-S||_p : S \in V_l^p\}. \tag{6}$$

Clearly we have the following equivalence:

$$\sigma_l^p(f) \sim ||f - P_l f||_p, \quad f \in L^p(\mathbb{R}^n) \ \ (1 \le p \le \infty).$$

Given $s > 0$ and $1 \le p, q \le \infty$. A function f is said to belong to $B_{pq}^s(\Phi)$ if

$$||f||_{B_{pq}^s(\Phi)} = ||f||_p + (\sum_{l=0}^\infty (2^{ls}\sigma_l^p(f))^q)^{1/q} < \infty \tag{7}$$

with the usual modification when $q = \infty$. Let

$$R_l f = P_{l+1} f - P_l f, \quad l = 0, 1, \dots . \tag{8}$$

We put

$$P_0 f(x) = \textstyle\sum_{j=1}^N \sum_{\nu\in\mathbb{Z}^n} a_{j0}(\nu)\phi_j(x-\nu),$$

$$R_l f(x) \ = \ \sum_{j=1}^N \sum_{\nu\in\mathbb{Z}^n} a_{j(l+1)}(\nu)\phi_j(2^{l+1}x-\nu). \tag{9}$$

Since Φ has stable shifts, we have

$$||P_0 f||_p \sim \sum_{j=1}^N ||a_{j0}||_{l^p},$$

$$||R_l f||_p \sim 2^{-(l+1)n/p} \sum_{j=1}^N ||a_{j(l+1)}||_{l^p}, \quad l = 0, 1, \dots . \tag{10}$$

Then for $f \in B^s_{pq}(\Phi)$ we have

$$f(x) = P_0 f(x) + \sum_{l=0}^{\infty} R_l f(x) \equiv \sum_{j=1}^N \sum_{l=0}^{\infty} \sum_{\nu \in \mathbb{Z}^n} a_{jl}(\nu) \phi_j(2^l x - \nu).$$

Moreover from [Wojtaszczyk, Theorem 5.10] there exists an associated set of wavelets $\{\psi_j^\epsilon\}_{j=1,\dots,N}^{\epsilon=1,\dots,2^n-1}$ in $\mathcal{L}^\infty$, that is, $\{\psi_j^\epsilon(x-\nu)\}_{j=1,\dots,N,\nu\in\mathbb{Z}^n}^{\epsilon=1,\dots,2^n-1}$ is an orthonormal basis in $W_0 = V_1^2 \ominus V_0^2$ in $L^2(\mathbb{R}^n)$, whose wavelet expansion of a function $f \in L^2(\mathbb{R}^n)$ is given by

$$f(x) = \sum_{j=1}^N \sum_{\nu \in \mathbb{Z}^n} b_j(\nu) \phi_j(x - \nu)$$
$$+ \sum_{j=1}^N \sum_{\epsilon=1}^{2^n-1} \sum_{l=0}^{\infty} \sum_{\nu \in \mathbb{Z}^n} b_{jl}^\epsilon(\nu) 2^{ln/2} \psi_j^\epsilon(2^l x - \nu) \tag{11}$$

where

$$b_j(\nu) = \langle f(y), \tilde{\phi}_j(y - \nu) \rangle, b_{jl}^\epsilon(\nu) = \langle f(y), 2^{ln/2} \psi_j^\epsilon(2^l y - \nu) \rangle. \tag{12}$$

Then we have

$$P_0 f(x) = \sum_{j=1}^N \sum_{\nu \in \mathbb{Z}^n} b_j(\nu) \phi_j(x - \nu),$$

$$R_l f(x) = \sum_{j=1}^N \sum_{\epsilon=1}^{2^n-1} \sum_{\nu \in \mathbb{Z}^n} b_{jl}^\epsilon(\nu) 2^{ln/2} \psi_j^\epsilon(2^l x - \nu), \quad l = 0, 1, \dots .$$

and

$$||R_l f||_p \sim 2^{ln(1/2-1/p)} \sum_{j=1}^N \sum_{\epsilon=1}^{2^n-1} ||b_{jl}^\epsilon||_{l^p} \quad (1 \le p \le \infty).$$

A following result can be proved from easy routine using Hardy's inequalty.

Theorem 3 ([Saka 1]). *Assume that a finite subset* $\Phi = \{\phi_1, \ldots, \phi_N\}$ *of* $\mathcal{L}^\infty$ *is refinable and has* L^2*-stable shifts. Given* $\alpha > 0$*, there are equivalences of the norm* $||f||_{B^\alpha_{pq}(\Phi)}$ *given in* (7)*, if one of them exists, for any* $1 \le p, q \le \infty$*, with the usual modification for* $q = \infty$*:*

(i) $||f||_p + (\sum_{l=0}^\infty (2^{l\alpha}||f - P_l f||_p)^q)^{1/q}$,

(ii) $||P_0 f||_p + (\sum_{l=0}^\infty (2^{l\alpha}||R_l f||_p)^q)^{1/q}$,

(iii) $(\sum_{l=0}^\infty (2^{l\alpha} 2^{-ln/p} \sum_{j=1}^N ||a_{jl}||_{l^p})^q)^{1/q}$,

where $\{a_{jl}\}$ *are given in* (9).

(iv) $\inf(\sum_{l=0}^\infty (2^{l\alpha} 2^{-ln/p} \sum_{j=1}^N ||c_{jl}||_{l^p})^q)^{1/q}$

where the infimum is taken over all admissible L^p*-convergent representations*

$f(x) = \sum_{j=1}^N \sum_{l=0}^\infty \sum_{\nu \in \mathbb{Z}^n} c_{jl}(\nu)\phi_j(2^l x - \nu)$,

(v) $\sum_{j=1}^N ||b_j||_{l^p} + (\sum_{l=0}^\infty (2^{l\alpha} 2^{ln(1/2-1/p)} \sum_{j=1}^N \sum_{\epsilon=1}^{2^n-1} ||b_{jl}^\epsilon||_{l^p})^q)^{1/q}$

where $\{b_j\}$ *and* $\{b_{jl}^\epsilon\}$ *are given in* (12).

Proposition 1 ([Saka 1]). *Given* $k + 1 > s > 0$*. Assume that* $\Phi = \{\phi_1, \ldots, \phi_N\}$ *of* $\mathcal{L}^\infty$ *is refinable and has* L^2*-stable shifts. Then we have for any* $1 \le p, q \le \infty$*,*

$$B^s_{pq}(\Phi) \subset B^s_{pq}(\mathbb{R}^n)$$

provided that there exists a positive number s_0 *with* $s_0 > s$ *such that* $\sup_{l \ge 0} 2^{ls_0} |\mathrm{osc}^k_p \phi_j(\cdot, l)|_p < \infty$ *for all* $j = 1, \ldots, N$*, where the norm* $|\cdot|_p$ *and* osc^k_p *are given in* (4) *and* (1) *respectively.*

Sketch of Proof. We shall prove for any $f \in B^s_{pq}(\Phi)$,

$$(\sum_{l=0}^\infty (2^{ls}\tilde{\sigma}^p_l(f))^q)^{1/q} \le C(||f||_p + (\sum_{l=0}^\infty (2^{ls}\sigma^p_l(f))^q)^{1/q})$$

where σ_l^p is the errors of L^p-approximation given in (6) associated with Φ and $\tilde{\sigma}_l^p(f) = ||\mathrm{osc}_p^k f(\cdot, l)||_p$. Since $\sigma_l^p(f) \to 0$ as $l \to \infty$ $(1 \le p \le \infty)$, we have an L^p-convergent series

$$
\begin{aligned}
f(x) &= P_0 f(x) + \sum_{l=0}^{\infty} R_l f(x) \\
&\equiv \sum_{j=1}^{N} \sum_{l=0}^{\infty} \sum_{\nu \in \mathbb{Z}^n} a_{jl}(\nu) \phi_j(2^l x - \nu)
\end{aligned}
$$

where $P_0 f(x) = \sum_{j=1}^N \sum_{\nu \in \mathbb{Z}^n} a_{j0}(\nu)\phi_j(x - \nu)$ and $R_l f(x) = \sum_{j=1}^N \sum_{\nu \in \mathbb{Z}^n} a_{j(l+1)}(\nu)\phi_j(2^{l+1}x - \nu)$ are given in (9). Then we have

$$
\begin{aligned}
\tilde{\sigma}_{l_0}^p(f) &= \tilde{\sigma}_{l_0}^p(P_0 f + \sum_{l=0}^{\infty} R_l f) \\
&\le \tilde{\sigma}_{l_0}^p(P_0 f) + \sum_{l=0}^{\infty} \tilde{\sigma}_{l_0}^p(R_l f) \equiv I_0 + \sum_{l=0}^{\infty} I_l'.
\end{aligned}
$$

We shall give an estimate of I_0. By (10) we have

$$
\begin{aligned}
I_0 &\le C \sum_{j=1}^{N} || \sum_{\nu \in \mathbb{Z}^n} |a_{j0}(\nu)| \mathrm{osc}_p^k \phi_j(x - \nu, l_0)||_p \\
&\le C \sum_{j=1}^{N} ||a_{j0}||_{l^p} |\mathrm{osc}_p^k \phi_j(\cdot, l_0)|_p \le C||P_0 f||_p \sup_j |\mathrm{osc}_p^k \phi_j(\cdot, l_0)|_p.
\end{aligned}
$$

If $l < l_0$, then we see by (10) that

$$
\begin{aligned}
I_l' &\le C 2^{-(l+1)n/p} \sum_{j=1}^{N} || \sum_{\nu} |a_{j(l+1)}(\nu)| \mathrm{osc}_p^k \phi_j(x - \nu, l_0 - l - 1)||_p \\
&\le C \sum_{j=1}^{N} 2^{-(l+1)n/p} ||a_{j(l+1)}||_{l^p} |\mathrm{osc}_p^k \phi_j(\cdot, l_0 - l - 1)|_p \\
&\le C||R_l f||_p \sup_j |\mathrm{osc}_p^k \phi_j(\cdot, l_0 - l - 1)|_p.
\end{aligned}
$$

If $l \geq l_0$, then we have by the definition,

$$I_l' \leq ||R_l f||_p.$$

From Hardy's inequality and Theorem 3, these complete the proof of Proposition 1.

A following corollary can be proved by the same way in the proof of Proposition 1.

Corollary. *Given $s > 0$. Assume that $\Phi = \{\phi_1, \ldots, \phi_N\}$ and $\Phi' = \{\phi_1', \ldots, \phi_L'\}$ of $\mathcal{L}^\infty$ are refinable and have L^2-stable shifts. Then we have for any $1 \leq p, q \leq \infty$,*

$$B_{pq}^s(\Phi') \subset B_{pq}^s(\Phi)$$

provided that there exists a positive number s_0 with $s_0 > s$ such that $\sup_{l\geq 0} 2^{ls_0}|\phi_j' - P_l\phi_j'|_p < \infty$ for all $j = 1, \ldots, L$, where the operator P_l is given in (5) associated with Φ.

For a positive integer k and $1 \leq p \leq \infty$, $\mathcal{L}_k^p = \mathcal{L}_k^p(\mathbb{R}^n)$ is denoted to be the space of all functions f such that $f(x)(1+|x|)^k \in \mathcal{L}^p$. If $\phi \in L^p(\mathbb{R}^n)$ $(1 \leq p \leq \infty)$ is compactly supported, then $\phi \in \mathcal{L}_k^p$. Furthermore, we observe that if there are constants $C > 0$ and $\delta > k$ such that $|\phi(x)| \leq C(1+|x|)^{-n-\delta}$ for all $x \in \mathbb{R}^n$ then $\phi \in \mathcal{L}_k^\infty$. For a finite subset Φ of $\mathcal{L}_k^\infty$, the domain of the operator P_l given in (5), can be extended to include the linear space $\mathbb{P}^k$ of all polynomials of degree no greater than k on $\mathbb{R}^n$.

For a finite subset Φ of $\mathcal{L}_k^1$, we say that Φ satisfies the Strang-Fix condition of order k if there is a finite linear combination ϕ of the functions of Φ and their shifts such that $\hat{\phi}(0) \neq 0$ and $\partial^\alpha\hat{\phi}(2\pi\nu) = 0,\ |\alpha| \leq k-1,\ \nu \in \mathbb{Z}^n$ with $\nu \neq 0$.

Lemma 1 ([Lei-Jia-Cheney]). *Let Φ be a finite subset of $\mathcal{L}_k^\infty$ that has L^2- stable shifts. Then Φ satisfies the Strang-Fix condition of order k if and only if $P_0 q = q$ for any $q \in \mathbb{P}^{k-1}$.*

Moreover, if this is the case, then we have

$$||P_l f - f||_p \leq C 2^{-lk} \sum_{|\alpha|=k} ||\partial^\alpha f||_p$$

for any f in the Sobolev space $W_k^p(\mathbb{R}^n)$ $(1 \leq p \leq \infty)$, with a constant C independent of f, p and l , that is, $W_k^p(\mathbb{R}^n) \subset B_{pq}^s(\Phi)$ if $0 < s < k$ and $1 \leq q \leq \infty$.

4 Characterization of Besov spaces by multiresolution approximation

Let $\Pi_l = \{2^{-l}(T+\nu)\}_{\nu\in\mathbb{Z}^n}$ for a nonnegative integer l.

Proposition 2 ([Saka 1]). *Given $1 \le p,\ q \le \infty$ and $k > s > 0$. Assume that a finite subset $\Phi = \{\phi_1, \dots, \phi_N\}$ of $\mathcal{L}_k^\infty$ satisfies*
(a) *Φ has L^2-stable shifts,*
(b) *Φ is refinable,*
(c) *Φ satisfies the Strang-Fix condition of order k.*
Then we have

$$B^s_{pq}(\mathbb{R}^n) \subset B^s_{pq}(\Phi)$$

.

Sketch of Proof. We shall prove for any $f \in B^s_{pq}(\mathbb{R}^n)$,

$$(\sum_{l=0}^{\infty}(2^{ls}\sigma_l^p(f))^q)^{1/q} \le C||f||_{B^s_{pq}(\mathbb{R}^n)}$$

where σ_l^p is given in (6) associated with Φ. We choose a function χ in $C_c^\infty(\mathbb{R}^n)$ such that $\int|\chi(u)|du = 1$ and supp $\chi \subset \{u \in \mathbb{R}^n : |u| < 1/k\}$. We write $\chi_l(u) = 2^{ln}\chi(2^l u)$, $\quad h_l(x) = \int(f(x) - \triangle_u^k f(x))\chi_l(u)du$ and $g_l = P_l h_l - h_l$ where P_l is given in (5) associated with Φ. Then we have for $1 \le p \le \infty$,

$$\begin{aligned} ||f - P_l f||_p &\le ||f-h_l||_p + ||g_l||_p + ||P_l h_l - P_l f||_p \\ &\le \ C||f-h_l||_p + ||g_l||_p \equiv CI_1 + I_2. \end{aligned}$$

Obviously we have :

$$I_1 \le C \sup_{|u|<2^{-l}} ||\triangle_u^k f||_p.$$

We shall give an estimate of I_2 :

$$\begin{aligned} I_2 &= (\textstyle\sum_{Q\in\Pi_l}\int_Q |g_l(x)|^p dx)^{1/p} \\ &= (\sum_{\nu\in\mathbb{Z}^n}\int_{2^{-l}T}|g_l(x-2^{-l}\nu)|^p dx)^{1/p}. \qquad (13) \end{aligned}$$

Let q_z be the $(k-1)$-th Taylor polynomial of h_l about $z \in \mathbb{R}^n$ and let r_z be the corresponding remainder. Since Φ satisfies the Strang-Fix condition of order k, we see from Lemma 1

$$\begin{aligned} g_l(x-2^{-l}\nu) &= P_l r_{x-2^{-l}\nu}(x-2^{-l}\nu) \\ &= 2^{ln}\int K(2^l x, 2^l y) r_{x-2^{-l}\nu}(y-2^{-l}\nu)dy \end{aligned}$$

where $K(x,y)=\sum_{j=1}^N \sum_{\nu\in\mathbb{Z}^n} \phi_j(x-\nu)\bar{\tilde{\phi}}_j(y-\nu)$. To estimate I_2, we use

$$\begin{aligned} & r_{x-2^{-l}\nu}(y-2^{-l}\nu) \\ &= \int_0^1 \sum_{|\beta|=k} \frac{k}{\beta!}\partial^\beta h_l(x+t(y-x)-2^{-l}\nu)(1-t)^{k-1}(y-x)^\beta dt, \end{aligned}$$

and

$$\begin{aligned} |\partial^\beta h_l(x)| \le C\sum_{e=1}^k &(\int_{|u|<1/k} |f(x-e2^{-l}u)|^p du)^{1/p} \\ \le\ & C\sum_{e=1}^k (2^{ln}\int_{|2^l u|<e/k} |f(x-u)|^p du)^{1/p} \\ \le\ & C2^{ln/p}(\int_{|2^l u|<1} |f(x-u)|^p du)^{1/p}. \end{aligned}$$

Hence we get an estimate:

$$
\begin{aligned}
&(\sum_{\nu\in\mathbb{Z}^n} |r_{x-2^{-l}\nu}(y-2^{-l}\nu)|^p)^{1/p} \\
&\leq \; C\int_0^1 \sum_{|\beta|=k}(\sum_{\nu}|\partial^\beta h_l(x+t(y-x)-2^{-l}\nu)|^p)^{1/p} \\
&\times \; (1-t)^{k-1}|x-y|^k dt \\
&\leq \; C\int_0^1 \sum_{|\beta|=k}(\sum_{\nu} 2^{ln}\int_{|2^l u|<1}|f(x+t(y-x)-2^{-l}\nu-u)|^p du)^{1/p} \\
&\times \; (1-t)^{k-1}|x-y|^k dt \\
&\leq \; C\int_0^1 2^{ln/p}(\sum_{\nu}\int_{2^{-l}(T+\nu)}|f(x+t(y-x)+u)|^p du)^{1/p} \\
&\times \; (1-t)^{k-1}|x-y|^k dt \\
&\leq \; C\int_0^1 2^{ln/p}||f||_p(1-t)^{k-1}|x-y|^k dt \leq C|x-y|^k 2^{ln/p}||f||_p.
\end{aligned}
$$

Hence, since $\Phi\subset\mathcal{L}_k^\infty$, we get an estimate of I_2 in (13):

$$
\begin{aligned}
I_2 &\leq C2^{ln}(\int_{2^{-l}T}\sum_{\nu}(\int|K(2^l x,2^l y)||r_{x-2^{-l}\nu}(y-2^{-l}\nu)|dy)^p dx)^{1/p} \\
&\leq \; C2^{ln}(\int_{2^{-l}T}(\int|K(2^l x,2^l y)|(\sum_{\nu}|r_{x-2^{-l}\nu}(y-2^{-l}\nu)|^p)^{1/p}dy)^p dx)^{1/p} \\
&\leq \; C2^{n(l+l/p)}||f||_p(\int_{2^{-l}T}(\int|K(2^l x,2^l y)||x-y|^k dy)^p dx)^{1/p} \\
&\leq \; C||f||_p(\int_T(\int|K(x,y)||2^{-l}(x-y)|^k dy)^p dx)^{1/p} \\
&\leq \; C||f||_p 2^{-lk}(\int_T(\int|K(x,y)||x-y|^k dy)^p dx)^{1/p} \leq C||f||_p 2^{-lk}.
\end{aligned}
$$

Now we combine the estimates of I_1 and I_2 to write

$$||f-P_l f||_p \leq CI_1+I_2 \leq C(\sup_{|2^l u|<1}||\triangle_u^k f||_p + 2^{-lk}||f||_p).$$

This implies that

$$(\sum_{l=0}^{\infty}(2^{ls}\sigma_l^p(f))^q)^{1/q} \leq C||f||_{B^s_{pq}(\mathbb{R}^n)}.$$

This completes the proof of Proposition 2.

A following theorem is an immediate consequence of Proposition 1 and Proposition 2. This theorem is a generalization of results in [Devore-Popov] and [Sickel].

Theorem 4 ([Saka 1]). *Given* $1 \le p,\ q \le \infty$ *and* $k > s > 0$. *Assume that a finite subset* $\Phi = \{\phi_1, \dots, \phi_N\}$ *of* $\mathcal{L}_k^\infty$ *satisfies*
(a) Φ *has* L^2*-stable shifts,*
(b) Φ *is refinable,*
(c) *there exists a positive number* s_0 *with* $s_0 > s$
such that $\sup_{l \ge 0} 2^{ls_0} |\mathrm{osc}_p^{k-1} \phi_j(\cdot, l)|_p < \infty$ *for all* $j = 1, \dots, N$,
(d) Φ *satisfies the Strang-Fix condition of order* k.
Then we have $B_{pq}^s(\mathbb{R}^n) = B_{pq}^s(\Phi)$ *with equivalent norms*

$$||f||_{B_{pq}^s(\mathbb{R}^n)} \sim ||f||_{B_{pq}^s(\Phi)}$$

where the norms $||f||_{B_{pq}^s(\mathbb{R}^n)}$ *and* $||f||_{B_{pq}^s(\Phi)}$ *are given in* (2) *and* (7) *respectively.*

Remark. When $\{\phi_j\}_{j=1}^N$ have compact supports, we see that the condition (c) in Theorem 4 can be rephrased as :

(c)$'$ There exists a positive number $s_0 > s$ such that

$$\sup_{l \ge 0} 2^{ls_0} ||\mathrm{osc}_p^{k-1} \phi_j(\cdot, l)||_p < \infty$$

for all $j = 1, \dots, N$, that is, $\phi_j \in B_{p\infty}^{s_0}(\mathbb{R}^n)$ if $s_0 < k$.

We say that a function on $\mathbb{R}^n$ is k-regular if it is of class C^k and rapidly decreasing in the sense that $|\partial^\alpha f(x)| \le C_N(1 + |x|)^{-N}$ for all $N = 0,\ 1,\ 2,\ \dots$ and all $|\alpha| \le k$. Any k-regular function belongs to $\mathcal{L}_N^\infty$ for any $N \ge 0$ and any k-regular function f satisfies the condition (c) in Theorem 4 : $\sup_{l \ge 0} 2^{lk} |\mathrm{osc}_p^{k-1} f(\cdot, l)|_p < \infty$.

Hence we get a result of [Wojtaszczk].

Corollary 1 . *Let* $1 \le p,\ q \le \infty$ *and* $k > s > 0$. *Assume that a finite subset* $\Phi = \{\phi_1, \dots, \phi_N\}$ *of* k*-regular functions on* $\mathbb{R}^n$ *satisfies:*
(a) Φ *has* L^2*-stable shifts,*
(b) Φ *is refinable.*

Then there exists a set $\{\psi_j^\epsilon\}_{j=1,\dots,N}^{\epsilon=1,\dots,2^n-1}$ *of* k*-regular wavelets associated with* Φ*, and we have equivalence of norms, if one of them exist, for wavelet expansion given in* (11) *with the usual modification for* $q=\infty$*:*

(i) $||f||_{B^s_{pq}(\mathbb{R}^n)}$,

$\sim$ (ii) $||f||_{B^s_{pq}(\Phi)}$,

$\sim$ (iii) $\sum_{j=1}^N ||b_j||_{l^p} + (\sum_{l=0}^\infty (2^{l(s+n/2-n/p)} \sum_{j=1}^N \sum_{\epsilon=1}^{2^n-1} ||b_{jl}^\epsilon||_{l^p})^q)^{1/q}$.

We define the tensor product B-spline by $\mathcal{M}_k = \prod_{i=1}^n M_k(x_i)$, $x = (x_1,\dots,x_n) \in \mathbb{R}^n$, $k = 1,2,\dots$. where $M_k(t)$ is the k-th order central B-spline, that is, $\hat{M}_k(t) = (\dfrac{\sin(t/2)}{t/2})^k$. Let us denote by $\{e^i\}_{i=1}^n$ the set of unit vectors in $\mathbb{R}^n$. We put $e^{n+1} = \sum_{i=1}^n e^i$, and $X = \{x^1,\dots,x^{d_0}\}$ with $x^1 = e^1,\dots,x^{d_1} = e^1, x^{d_1+1} = e^2,\dots,x^{d_1+d_2} = e^2,\dots,x^{d_1+\dots+d_n+1} = e^{n+1},\dots,x^{d_0} = e^{n+1}$ where $d_0 = d_1+\dots+d_{n+1}$.

We denote the box spline $B(x,X)$ corresponding to X given by $\hat{B}(x,X) = (2\pi)^{-n/2}\prod_{j=1}^{d_0} \dfrac{1-e^{ix^j\cdot x}}{ix^j\cdot x}$. In the case , the k-th order tensor product B-spline $\mathcal{M}_k$ satisfies the conditions of Theorem 4, particularly, $\mathcal{M}_k \in B^{k-1+1/p}_{p\infty}(\mathbb{R}^n)$ and $\mathcal{M}_k$ satisfies the Strang-Fix condition of order k.

The above box spline $B(x,X)$ also satisfies the conditions of Theorem 4 replacing the above k by $k = \min\{d_i+d_j : \ i,j = 1,\dots,n+1, \ i \neq j\}$.

Hence we get results of [DeVore-Popov] and [Sickel].

Corollary 2 . *Theorem 4 remains true for the tensor product B-spline* $\Phi = \{\mathcal{M}_k\}$ *or the box spline* $\Phi = \{B(x,X)\}$.

A following theorem is a pointwise version of Corollary 1 of Theorem 4.

Theorem 5 ([Saka 1]). *Suppose that* $k > s > 0$*. Assume that a finite subset* $\Phi = \{\phi_1,\dots,\phi_N\}$ *of* k*-regular functions on* $\mathbb{R}^n$ *satisfies:*

(a) Φ *has* L^2*-stable shifts,*

(b) Φ *is refinable.*

Then for $x \in \mathbb{R}^n$ *and a bounded function* f *on* $\mathbb{R}^n$ *, following statements are equivalent:*

(i) $f \in C^s(x)$,

(ii) $|f(y) - P_l f(y)| \le C(2^{-l} + |x-y|)^s \quad l = 0,1,2,\cdots$

where $P_l f$ *is given in* (5).

Corollary. *Suppose that the conditions in Theorem 5 are satisfied. Let* $s > s' > 0$.

(a) *If* $f \in C^s(x)$, *we have*

$$|R_l f(y)| \leq C(2^{-l} + |x-y|)^s \quad l = 0, 1, 2, \ldots$$

where $R_l f$ *is given in* (8). *If it holds*

$$|R_l f(y)| \leq C 2^{-sl}(1 + 2^l|x-y|)^{s'} \quad l = 0, 1, 2, \ldots,$$

then $f \in C^s(x)$.

(b) *If* $f \in C^s(x)$, *we have*

$$|b^{\epsilon}_{jl}(\nu)| \leq C 2^{-(s+\frac{n}{2})l}(1 + |2^l x - \nu|)^s$$

for $j = 1, \ldots, N, l = 1, 2, 3, \ldots, \epsilon = 1, \ldots, 2^n - 1$ *and any* $\nu \in \mathbb{Z}^n$ *where* $b^{\epsilon}_{jl}(\nu)$ *is given in* (12). *If it holds*

$$|b^{\epsilon}_{jl}(\nu)| \leq C 2^{-(s+\frac{n}{2})l}(1 + |2^l x - \nu|)^{s'}$$

for $j = 1, \ldots, N, \quad l = 1, 2, 3, \ldots$ and $\epsilon = 1, \ldots, 2^n - 1$ *and any* $\nu \in \mathbb{Z}^n$, *then* $f \in C^s(x)$.

(c) *For* $\{a_{jl}(\nu)\}$ *given in* (9), *if it holds*

$$|a_{jl}(\nu)| \leq C 2^{-sl}(1 + |2^l x - \nu|)^{s'}$$

for $j = 1, \ldots, N, \quad l > 0$ and $\nu \in \mathbb{Z}^n$, *then* $f \in C^s(x)$.

5 Scaling exponents

For $1 \leq p, q \leq \infty$ we define $\alpha_{pq}(f) = \sup\{s \geq 0 : f \in B^s_{pq}(\mathbb{R}^n)\}$ for functions $f \in L^p(\mathbb{R}^n)$ and $\alpha_{pq}(f, D) = \sup\{s \geq 0 : f \in B^s_{pq}(D)\}$ for a domain D in $\mathbb{R}^n$. If there is not a positive number s with $f \in B^s_{pq}(\mathbb{R}^n)$, then we define $\alpha_{pq}(f) = 0$. We remark that $\alpha_{pq}(f) > 0$ for any $f \in L^p(\mathbb{R}^n)$ in the case $1 \leq p < \infty$.

In the same manner we define $\alpha_{pq}(f, x) = \sup\{s \geq 0 : f \in T^s_{pq}(x)\}$ for $x \in \mathbb{R}^n$ and bounded functions f on $\mathbb{R}^n$. We put $\alpha_p(f) = \alpha_{p\infty}(f)$, $\alpha(f) = \alpha_\infty(f)$, $\alpha_p(f, x) = \alpha_{p\infty}(f, x)$ and $\alpha(f, x) = \alpha_\infty(f, x)$.

We can prove a following proposition by the embedding theorem.

Proposition 3 ([Saka 1]).

(i) $\alpha_p(f) = \alpha_{p\eta}(f)$ *for* $1 \leq p, \eta \leq \infty$,

(ii) $\alpha(f) > \alpha_p(f) - \frac{n}{p} \geq \alpha_q(f) - \frac{n}{q}$ *for* $1 \leq q \leq p < \infty$,

(iii) $\alpha_p(f, x) = \alpha_{p\eta}(f, x)$ *for* $1 \leq p, \eta \leq \infty$,

(iv) $\alpha(f) \leq \alpha(f, x) \leq \alpha_p(f, x) \leq \alpha_q(f, x)$ *for* $1 \leq q \leq p < \infty$.

Let ϕ_1 be the Rademacher function given by $\phi_1(x) = 1$ $(0 \leq x < 1/2)$, $\phi_1(x) = -1$ $(1/2 \leq x < 1)$, $\phi_1(x) = 0$ elsewhere.

Let ϕ_2 be the tent function given by $\phi_2(x) = 2x$ $(0 \leq x < 1/2)$, $\phi_2(x) = 2(1-x)$ $(1/2 \leq x < 1)$, $\phi_2(x) = 0$ elsewhere .

For $\beta > 0$, we conside a function F which is given by a series

$$F(x) = \sum_{l=0}^{\infty} \sum_{\nu=0}^{2^l-1} 2^{-\beta l} \phi_k(2^l x - \nu)$$

.

Theorem 6 .

Let $1 \leq p \leq \infty$ *and* $k = 1, 2$.

(a) *If* $\beta < k - 1 + \frac{1}{p}$, *then* $\alpha_p(F) = \beta$,

(b) *If* $k - 1 + \frac{1}{p} \leq \beta$, *then* $\alpha_p(F) = k - 1 + \frac{1}{p}$.

Sketch of Proof.

(I) The case $k = 1$.

Let

$$F(x) = \sum_{l=0}^{\infty} \sum_{\nu=0}^{2^l-1} 2^{-\beta l} \phi_1(2^l x - \nu). \tag{14}$$

Step (i)

If $p = \infty$, we should delete the statement (a).

Step(ii)

When $\beta = 1$, we get $F(x) = 2(1-2x)$ on $I = [0,1]$, $\quad F(x) = 0, \ \ x \notin I$. Hence $\alpha_p(F) = \frac{1}{p} \ \ (1 \leq p \leq \infty)$.

Step (iii)

If $p = \infty$ and $\beta \neq 1$, then $F \notin C^\alpha(\mathbb{R}) = B^\alpha_{\infty\infty}(\mathbb{R})$ for each $\alpha > 0$ because of that F is not continuous. Hence $\alpha_\infty(F) = 0$ if $\beta \neq 1$.

Step (iv)

Suppose that $\beta \neq 1$ and $1 \leq p < \infty$. Then the function F is discontinuous at dyadic points with jumps and so we have from the embedding theorem(cf. [Triebel]) $F \notin B^s_{p\infty}(\mathbb{R})$ if $s \geq \frac{1}{p}$. Hence we have

$$\alpha_p(F) \leq \frac{1}{p}. \tag{15}$$

We can consider the expansion (14) as a wavelet expansion on $\mathbb{R}$. Then from Theorem 4, we see if $0 < s < \frac{1}{p}$, $F \in B^s_{p\infty}(\mathbb{R})$ if and only if $s \leq \beta$. Therefore if $\beta < \frac{1}{p}$, then $\alpha_p(F) = \beta$, and if $\beta \geq \frac{1}{p}$, then $\alpha_p(F) \geq \frac{1}{p}$. Hence by (15) we have $\alpha_p(F) = \frac{1}{p}$ if $\beta \geq \frac{1}{p}$. This completes the case $k = 1$.

(II) The case $k = 2$.

Let

$$F(x) = \sum_{l=0}^{\infty} \sum_{\nu=0}^{2^l-1} 2^{-\beta l} \phi_2(2^l x - \nu)$$

.

Step (i)

When $\beta \neq 2$ and $\beta > 1$, then we have $\alpha_p(F) = \alpha_p(F') + 1$ where $F'(x) = \sum_{l=0}^{\infty} \sum_{\nu=0}^{2^l-1} 2^{-(\beta-1)l} \phi_1(2^l x - \nu)$.

From (I), we obtain $\alpha_p(F) = \beta$ if $1 < \beta < 1 + \frac{1}{p}$ and $\alpha_p(F) = 1 + \frac{1}{p}$ if $\beta \geq 1 + \frac{1}{p}$.

Step (ii)

When $\beta < 1$, we apply wavelet analysis. We define a continuous wavelet transform by

$$Wf(x,t) = \int f(y) t^{-n} \varphi(\frac{y-x}{t}) dy$$

where $\varphi \in C^m(\mathbb{R}^n)$ with compact support such that $\int x^\alpha \varphi(x)dx = 0$ for each $0 \leq |\alpha| \leq m$. Since we have as the continuous version for wavelet characterization of Besov spaces that

$$||f||_{B^s_{pq}(\mathbb{R})} = ||f||_p + (\int_0^1 (t^{-s}||Wf(\cdot,t)||_p)^q \frac{dt}{t})^{1/q}$$

where $Wf(x,t)$ is a wavelet transform of f, then we see

$$\begin{aligned}\alpha_p(f) &= \sup\{s \geq 0 : \limsup_{t\to 0} t^{-s}||Wf(\cdot,t)||_p < \infty\} \\ &= \inf\{s \geq 0 : \limsup_{t\to 0} t^{-s}||Wf(\cdot,t)||_p > 0\}.\end{aligned}$$

We can get following estimates by a method of [Jaffard 2].

Lemma A .
Let $1 \leq p \leq \infty$. If $\beta < 1$, then

(a) $\limsup_{N\to\infty} 2^{\beta N}(\int |WF(x,2^{-N})|^p dx)^{1/p} < \infty$,

(b) $\limsup_{N\to\infty} 2^{\beta N}(\int |WF(x,2^{-N})|^p dx)^{1/p} > 0$

From this lemma we get $\alpha_p(F) = \beta$ if $\beta < 1$.

Step (iii)
In the case $\beta = 1$ we use a following lemma

Lemma B (cf. [Jonsson-Kamont]).
Let $1 \leq p \leq \infty$ and $1/p < s < 1$. Let $F(x) = \sum_{l=0}^\infty \sum_{\nu=0}^{2^l-1} 2^{-\beta l}\phi_2(2^l x - \nu)$
Then $F \in B^s_{p\infty}(\mathbb{R})$ if and only if $s \leq \beta$.

From this lemma , if $\beta = 1$, then $\alpha_p(F) \geq 1$. If $\alpha_p(F) > 1$, then there is $s > 1$ such that $F \in B^s_{p\infty}(\mathbb{R})$. Hence $F' \in B^{s-1}_{p\infty}(\mathbb{R})$ but

$$F(x) = \sum_{l=0}^\infty \sum_{\nu=0}^{2^l-1} 2^{-l}\varphi_2(2^l x - \nu)$$

is nondifferentiable at evry point $x \in I$. This is a contradiction. Hence $\alpha_p(F) = 1$. This completes the case $k = 2$.

6 Estimates of pointwise regularity

Let $T = [0,1]^n$ and $\Pi_l = \{2^{-l}(T+\nu)\}_{\nu\in\mathbb{Z}^n}$ for a nonnegative integer l. We write $Q = 2^{-l}(T+\nu_Q)$ for $Q \in \Pi_l$. Let $\Pi_l(T) = \{Q \in \Pi_l : \ Q \subset T\}$ and $\Pi(T) = \cup_{l=0}^{\infty}\Pi_l(T)$. We put $\Pi_1 = \{2^{-1}(T+\gamma_i), i = 1, 2, \ldots, 2^n\}$ and $\Gamma_0 = \{\gamma_1, \cdots, \gamma_{2^n}\}$. Then for $Q \in \Pi_l(T)$, ν_Q is of a form $\nu_Q = 2^{l-1}\gamma_{i_1} + \cdots + \gamma_{i_l}$, $\gamma_{i_1}, \cdots, \gamma_{i_l} \in \Gamma_0$ and we write $M_Q y = 2^l y - \nu_Q$ and $\mu_Q = \mu_{i_1}\cdots\mu_{i_l}$ for $l > 0$ where $\mu_1, \mu_2, \ldots, \mu_{2^n}$ are real or complex numbers with $0 < |\mu_i| < 1$, $i = 1, \ldots, 2^n$. For $l = 0$ we put $\mu_T = 1$. Let ϕ be a bounded function which is zero outside T^o, the interior of T.

We consider a bounded function f which is given by a series

$$f(y) = \sum_{Q\in\Pi(T)} \mu_Q \phi(M_Q y), \quad y \in \mathbb{R}^n. \tag{16}$$

We remark that $\alpha(f) \le \alpha(\phi)$. Let

$$\begin{aligned}\tau_0(x) &\equiv \liminf_{l\to\infty} \inf_{K_l(x)\ni Q} \frac{\log|\mu_Q|}{\log(2^{-l} + |x - 2^{-l}\nu_Q|)} \\ &= \liminf_{l\to\infty} \inf_{K_l(x)\ni Q} \frac{\log|\mu_Q|}{\log 2^{-l}}\end{aligned}$$

where $K_l(x) \equiv \{Q \in \Pi_l(T) : \ B(x, 2^{-l}) \cap Q \neq \emptyset\}$ and $B(x, 2^{-l})$ is a ball centered at x with a radius 2^{-l}. When $x \in \Omega \equiv \cap_{l=0}^{\infty} \cup_{Q\in\Pi_l(T)} Q^o$(the interior of Q) there exists a unique sequence $\{Q_{l,x}\}_{l\ge 0}$ such that $Q_{l,x} \in \Pi_l(T)$ and $x \in Q^o_{l,x}$. Then we have for $x \in \Omega$

$$\tau_0(x) = \liminf_{l\to\infty} \frac{\log|\mu_{Q_{l,x}}|}{\log 2^{-l}}.$$

Let for $x \in \Omega$

$$\tau_1(x) \equiv \liminf_{l\to\infty} \frac{\log|\mu_{Q_{l,x}}|}{\log \Delta_l(x)}$$

where $\Delta_l(x) = \mathrm{dist}(x, \partial Q_{l,x})$ is the distance from x to the boundary $\partial Q_{l,x}$ of $Q_{l,x}$. We remark for $x \in \Omega$, $\tau_0(x) = \tau_1(x)$ if $\sup_{l\ge 0} \dfrac{\Delta_l(x)}{\Delta_{l+1}(x)} < \infty$.

A following theorem may be proved by the same way as in [Saka 2].

Theorem 7 . *Let f and ϕ be bounded functions given in* (16). *Then we have*

(i) $\alpha(f,x) \geq \min(\alpha(\phi), \tau_0(x))$ *for* $x \in T$,

(ii) $\alpha(f,x) \geq \min_i(\alpha(\phi,\Omega_i), \tau_1(x))$
for $x \in \Omega$ *with* $\sup_{l\geq 0} \dfrac{\Delta_l(x)}{\Delta_{l+1}(x)} < \infty$ *where* $\Omega_i \equiv 2^{-1}(T^o + \gamma_i)$, $\gamma_i \in \Gamma_0$, $i = 1, \dots, 2^n$ *and* $\alpha(\phi, \Omega_i) = \sup\{s \geq 0 : \phi \in C^s(\Omega_i)\}$ *and* $C^s(\Omega_i)$ *is defined as the Besov space* $B^s_{\infty\infty}(\Omega_i)$ *on* Ω_i.

(iii) *Suppose that* $\phi \in C^\infty(\Omega_i)$, $i = 1, \dots, 2^n$ *and there exist a positive number* s_0 *and* $y_0 \in T^o$ *such that*

$$\sup_{l\geq 0} \sup_y \frac{|f_l(y)|}{(2^{-l} + |y - y_0|)^{s_0}} = \infty.$$

Then $\tau_0(x) \geq \alpha(f,x)$ *for* $x \in T$ *where* f_l *is given in* (3).

Corollary. *Let ϕ be a bounded function on $\mathbb{R}^n$ such that $\phi \in C^\infty(\Omega_j)$, $j = 1, \dots, 2^n$ and $\phi = 0$ outside T^o. Consider a bounded function f given by* (16) *satisfying the condition* (iii) *in Theorem 7. Then we have*

(i) $\tau_0(x) \geq \alpha(f,x) \geq \min(\alpha(\phi), \tau_0(x)), \quad x \in T$,

(ii) *for x in Ω with* $\sup_{l\geq 0} \dfrac{\Delta_l(x)}{\Delta_{l+1}(x)} < \infty$, $\alpha(f,x) = \tau_0(x) = \tau_1(x)$.

Examples.

We put $\Pi = \{T + \nu\}_{\nu\in\mathbb{Z}}$ with the interval $T = [0,1]$ on $\mathbb{R}$.

(a) We consider the Takagi function such that

$$f(x) = \sum_{l=0}^{\infty} \sum_{Q\in\Pi_l(T)} \mu^l \phi(M_Q x), \quad \forall x \in \mathbb{R}$$

where $0 < \mu < 1$ and ϕ is a bounded function such that $\phi(x) = x$ $(0 < x \leq \frac{1}{2})$, $\phi(x) = 1 - x$ $(\frac{1}{2} \leq x < 1)$, $\phi(x) = 0$ (otherwise).

Let $\tau = \dfrac{\log \mu}{\log 2^{-1}}$. Then from the corollary of Theorem 7, if $\tau \leq 1$, $\tau = \alpha(f,x)$ for each $x \in T$.

(b) We consider the Weierstrass function

$$f(x) = \sum_{l=0}^{\infty} \mu^l \phi(2^l x)$$

with $0 < \mu < 1$ and $\phi(x) = \sin 2\pi x \quad (x \in \mathbb{R})$. The proof of Theorem 7 can be also applied to this function case.

Then we have $\tau = \alpha(f, x), \quad \forall x \in \mathbb{R}$. where the constant $\tau = \dfrac{\log \mu}{\log 2^{-1}}$ is given in the part (a) above.

(c) We consider Lèvy's function

$$f(x) = \sum_{l=0}^{\infty} \sum_{Q \in \Pi_l(T)} 2^{-l} \phi(M_Q x), \quad \forall x \in \mathbb{R}$$

where $\phi(x) = x - \dfrac{1}{2} \quad (0 < x < 1), \quad \phi(x) = 0 \quad$ (otherwise).

Then we can see that $1 = \tau_1(x) = \alpha(f, x)$ for a point x in Ω with $\sup_{l \geq 0} \dfrac{\Delta_l(x)}{\Delta_{l+1}(x)} < \infty$.

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