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THE SPECTRUM OF THE LAPLACIAN ACTING ON 2-FORMS AND CURVATURE OF KÄHLERIAN MANIFOLD

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Abstract:

In [1], we studied the effect of the spectrum of Laplacian acting on 2–forms of a Riemannian manifold. Now we consider complex Kählerian manifolds as a space. We get the result that a Kählerian manifold, whose spectrum of 2–forms is equal to that of the complex projective space, must be the complex projective space under the condition that its complex dimension runs 3, 4, 7∼94.

1. Introduction and statement of result. By $Spec^p\ (M, g)$ we denote the sequence of eigen–values of the Laplacian acting on p–forms of a Riemannian manifold (M, g). In [1], we studied the effect of $Spec^2\ (M, g)$ of Riemannian manifold (M, g). There are several studies about the relations between the spectrum $Spec^0\ (M, g, J)$ or $Spec^1\ (M, g, J)$ and the curvature of Kählerian manifold (M, g, J). Let n be the complex dimension of M and m be the real dimension of M, i. e. $m = 2n$.

THEOREM A. (S. TANNO [4]) Let (M, g, J) be a compact connected Kählerian manifold, $m = 2n \leqq 12$. Let $(CP^n(H), g_0, J_0)$ be a complex n–dimensional projective space with the Fubini–Study metric of constant holomorphic sectional curvature H.

If $Spec^0\ (M, g, J) = Spec^0\ (CP^n(H), g_0, J_0)$, then (M, g, J) is holomorphically isometric to $(CP^n(H), g_0, J_0)$.

THEOREM B. (S. TANNO [5]) Let (M, g, J) and (M', g', J') be compact connected Kählerian manifolds with $Spec^1\ (M, g, J) = Spec^1\ (M', g', J')$.

For $16 \leqq m =$ real dim. $M \leqq 102$, (M, g, J) is of constant holomorphic sectional curvature H, if and only if (M', g', J') is of constant holomorphic sectional curvature $H' = H$.

In this paper we study the effect of $Spec^2\ (M, g, J)$ of a Kählerian manifold (M, g, J). The results obtained are following

THEOREM. *Let (M, g, J) and (M', g', J') be compact connected Kählerian manifolds. We assume that $Spec^2(M, g, J) = Spec^2(M', g', J')$ holds good. Then, for m (= real dim. M) $=6, 8$ or $14 \sim 188$, (M, g, J) is of constant holomorphic sectional curvature H if and only if (M', g', J') is of constant holomorphic sectional curvatue $H' = H$.*

COROLLARY. *The complex projective space (CP^n, g_0, J_0) with the Fubini-Study metric, $n = 3, 4$ or $7 \sim 94$, is completely characterized by the spectrum of the Laplacian acting on 2-forms.*

2. Preliminaries. Let (M, g, J) be a Kählerian manifold with almost complex structure tensor J and Kählerian metric tensor g. They satisfy

$$(2.1) \qquad J^i{}_j J^j{}_k = -\delta^i{}_k \text{ and } \qquad g_{ij} J^i{}_r J^j{}_s = g_{rs}.$$

By $R = (R^i{}_{jkl})$, $\rho = (\rho_{jk}) = (R^l{}_{jkl})$ and τ, we denote the Riemannian curvature tensor, the Ricci curvature tensor and the scalar curvature, respectively.

A Kählerian manifold (M, g, J), $m \geqq 4$ is of constant holomorphic sectional curvature H, if and only if

$$(2.2) \qquad R_{ijkl} = \frac{H}{4}(g_{jk}g_{il} - g_{jl}g_{ik} + J_{jk}J_{il} - J_{jl}J_{ik} - 2J_{ij}J_{kl})$$

holds. Then ρ_{jk} and τ are given by

$$(2.3) \qquad \rho_{jk} = \frac{m+2}{4} H g_{jk} \quad , \qquad \tau = \frac{m(m+2)}{4} H.$$

The Bochner curvature tensor $B = (B^i{}_{jkl})$, $B_{ijkl} = g_{ir}B^r{}_{jkl}$, is given by (cf. [3])

$$(2.4) \qquad B_{ijkl} = R_{ijkl} - \frac{1}{m+4}(\rho_{jk}g_{il} - \rho_{jl}g_{ik} + g_{jk}\rho_{il} - g_{jl}\rho_{ik}$$
$$+ \rho_{jr}J^r{}_k J_{jl} - \rho_{jr}J^r{}_l J_{ik} + J_{jk}\rho_{ir}J^r{}_l - J_{jl}\rho_{ir}J^r{}_k$$
$$- 2\rho_{kr}J^r{}_l J_{ij} - 2\rho_{ir}J^r{}_j J_{kl}) + \frac{1}{(m+2)(m+4)}(g_{jk}g_{il} - g_{jl}g_{ik}$$
$$+ J_{jk}J_{il} - J_{jl}J_{ik} - 2J_{kl}J_{ij})\tau$$

$B(g) = |B|^2 = \langle B, B \rangle$ is given by (cf. [3])

$$(2.5) \qquad B(g) = |R|^2 - \frac{16}{m+4}|\rho|^2 + \frac{8}{(m+2)(m+4)}\tau^2.$$

$$(2.6) \qquad G(g) = |\rho|^2 - \frac{1}{m}\tau^2.$$

A Kählerian manifold (M, g, J) is of constant holomorphic sectional curvature if and only if $B(g) = 0$ and $G(g) = 0$.

3. Proof of the theorem. We use the same notation in [1].

(3.1) $Spec^p\ (M, g, J) = \{\ 0 \geqq \lambda_{1,p} \geqq \lambda_{2,p} \geqq \lambda_{3,p} \geqq \cdots\cdots \geqq \lambda_{k,p} \geqq \cdots\cdots\cdots \downarrow -\infty\ \}$

We use the Minakshisundaram–Pleijel–Gaffney's asymptotic expansion

(3.2) $$\sum_{k=1}^{\infty} exp\,(\lambda_{k,p} t) \underset{t \downarrow 0}{\sim} (4\pi t)^{-\frac{m}{2}}\ (a_{0,p} + a_{1,p} t + a_{2,p} t^2 + \cdots\cdots\cdots\cdots\cdots).$$

It may be noticed that instead of $Spec^2\ (M, g, J)$ we use olny $a_{k,2}$ for $k = 0, 1, 2.$ $a_{0,p}$, $a_{1,p}$ and $a_{2,p}$ in (3.2) were calculated by V.K. Patodi [2].

(3.3) $$a_{0,2} = \binom{m}{2} \int_M dM$$

(3.4) $$a_{1,2} = \int_M \frac{m^2 - 13m + 24}{12} \tau dM$$

and

(3.5) $$a_{2,2} = \int_M [C_1 (m, 2)\ \tau^2 + C_2 (m, 2)\ |\ \rho\ |^2 + C_3 (m, 2)\ |\ R\ |^2]\ dM$$

where

(3.6) $$C_1 (m, 2) = \frac{m^2 - 25m + 120}{144}$$

(3.7) $$C_2 (m, 2) = \frac{-m^2 + 181m - 1080}{360}$$

and

(3.8) $$C_3 (m, 2) = \frac{m^2 - 31m + 240}{360}.$$

REMARK; $Spec^2\ (M, g, J) = Spec^2\ (M', g', J')$ implies

(i) $m = m'$

(ii) volume of M = volume of M'.

By (2.5) and (2.6), $|\ R\ |^2$ and $|\ \rho\ |^2$ are written by $B(g)$, $G(g)$ and τ^2 and we get

(3.9) $$a_{2,2} = \int_M \{\ [C_1 (m, 2) + \frac{C_2 (m, 2)}{m} + \frac{8C_3 (m, 2)}{m (m+2)}\ ]\ \tau^2 + C_3(m, 2) B(g)$$

$$+ [C_2 (m, 2) + \frac{16}{m+4} C_3 (m, 2)]\ G(g)\}\ dM$$

We denote the coefficients of τ^2 and $G(g)$ by $\Psi(m)$ and $\Phi(m)$ respectively, i. e.

(3.10) $$\Phi (m) = C_2 (m, 2) + \frac{16}{m+4} C_3 (m, 2)$$

$$(3.11)\quad \Psi(m) = C_1(m,2) + \frac{1}{m} C_2(m,2) + \frac{8}{m(m+2)} C_3(m,2).$$

Then the following (3.12) is directly derived from (3.6), (3.7), (3.8), (3.10) and (3.11)

$$(3.12)\quad \begin{aligned} \Phi(m) &= \frac{1}{360}(-m^3 + 193m^2 - 852m - 480) \\ \Psi(m) &= \frac{1}{720}(5m^4 - 117m^3 + 724m^2 - 732m - 480) \end{aligned}$$

From (3.1) ~ (3.5) the condition $Spec^2(M, g, J) = Spec^2(M', g', J')$ implies

$$(3.13)\quad \int_M dM = \int_{M'} dM' ,$$

$$(3.14)\quad \int_M \tau dM = \int_{M'} \tau' dM' ,$$

and

$$(3.15)\quad \int_M [C_3(m,2) B(g) + \Phi(m) G(g) + \Psi(m)\tau^2]\, dM$$

$$= \int_{M'} [C_3(m,2) B(g') + \Phi(m) G(g') + \Psi(m)\tau'^2]\, dM' .$$

Now we assume that (M, g, J) is of constant holomrphic sectional curvature. Then $B(g) = G(g) = 0$ hold on M and τ is constant on M. Therefore by (3.15), we get

$$(3.16)\quad \int_{M'} [C_3(m,2) B(g') + \Phi(m) G(g')]\, dM' = \Psi(m) [\int_M \tau^2 dM - \int_{M'} \tau'^2 dM'].$$

By using the Schwarz's inequality for τ', we get

$$(3.17)\quad \int_{M'} \tau'^2\, dM' \geqq \frac{[\int_{M'} \tau'\, dM']^2}{\int_{M'} dM'}$$

where the equality holds if and only if τ' is constant on M'. On the other hand, by (3.13), (3.14) and the fact that τ is constant on M, the right-hand side of (3.17) is $\int_M \tau^2\, dM$. So we get the inequality

$$(3.18)\quad \int_M \tau^2\, dM - \int_{M'} \tau'^2\, dM' \leqq 0$$

where the equality holds if and only if τ' is constant and equal to τ.

Therefore $B(g') = 0$, $G(g')=0$ and $\tau' = $ constant $(= \tau)$ hold for m such that

$$(3.19)\qquad m = 2n \geqq 4,\ C_3(m, 2) > 0,\ \Phi(m) > 0 \text{ and } \Psi(m) \geqq 0.$$

We see easily that m which satisfies (3.19) runs 6, 8 or 14 ~ 188. $B(g') = 0$ and $G(g') = 0$ hold simultaneously if and only if (M', g', J') is of constant holomorphic sectional curvature. Thus the theorem is completely proved.

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