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THE SPECTRUM OF THE LAPLACIAN ACTING ON 2-FORMS AND CURVATURE OF RIEMANNIAN MANIFOLD

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Abstract:

We study in this paper how the spectrum of the Laplace–Beltrami operator acting on 2–forms determines curvature of a Riemannian manifold. We see that if the spectrum of 2–forms of an arbitrary Riemannian manifold is equal to the standard sphere then the preceeding manifold is the standard sphere when its dimension runs 2, 3, 6, 7, 8, 14, 17∼178.

1. Introduction and statement of result; Let (M,g) be a compact connected orientable Riemannian manifold with a Riemannian metric g. The dimension of M is denoted by m. Let $\triangle$ be the Laplace–Beltrami operator acting on exterior p–forms on M $(0\leqq p\leqq m)$. The Laplacian $\triangle$ acting on p–forms has an infinite sequence

$$(1.1)\qquad 0 \geqq \lambda_{1,p} \geqq \lambda_{2,p} \geqq \lambda_{3,p} \geqq \lambda_{4,p} \geqq\cdots\cdots\geqq \lambda_{k,p} \geqq\cdots\cdots \downarrow -\infty$$

of eigen–values each eigen–value being repeated as many times as its multiplicity indicates. By $Spec^p\ (M, g)$, we denote the sequence (1.1). There are several results about the problem; how the $Spec^p\ (M, g)$ determines the structure of (M, g).

THEOREM A. (M. BERGER [1]) Let (M, g) and (M', g') be compact connected orientable Riemannian manifolds. Assume that $Spec^0\ (M, g) = Spec^0\ (M', g')$ holds good. Then, for $m = 2, 3, 4$, (M, g) is of constant curvature c if and only if (M', g') is of constant curvature c, where the condition that $\chi(M) = \chi(M')$ holds is added for $m=4$. $\chi(M)$ denotes the Euler–Poincaré characteristic of M.

THEOREM B. (T. SAKAI [5]) Let (M, g) and (M', g') be compact connected orientable Einstein manifolds with dimension 6. Assume that $\chi(M) = \chi(M')$ and $Spec^0\ (M, g) = Spec^0\ (M', g')$ hold. Then (M, g) is locally symmetric if and only if (M', g') is locally symmetric.

THEOREM C. (V. K. PATODI [4]) Let (M, g) and (M', g') be compact connected orientable Riemannian manifolds. Assume that $Spec^p(M, g) = Spec^p(M', g')$ hold for $p = 0$ and $p = 1$. Then (M, g) is of constant curvature c if and only if (M', g') is of constant curvature c and (M, g) is an Einstein space if and only if (M', g') is an Einstein space for every m.

THEOREM D. (S. TANNO [7]) Let (M, g) and (M', g') be compact connected orientable Riemannian manifolds. Assume that $Spec^0(M, g) = Spec^0(M', g')$. Then, for $2 \leqq m \leqq 5$, (M, g) is of constant curvature c, if and only if (M', g') is of constant curvature $c' = c$.

REMARK: THEOREM D is a generalization of THEOREM A.

THEOREM E. (K. II [2]) Let (M, g) and (M', g') be compact connected orientable Riemannian manifolds. Assume that $Spec^p(M, g) = Spec^p(M', g')$ hold for $p = 0, 1$. If (M, g) is a locally symmetric Einstein space, then (M', g') is also locally symmetric.

THEOREM F. (S. TANNO [8]) Let (M, g) and $(M', g',)$ be compact connected orientable Riemannian manifolds. Assume that $Spec^1(M, g) = Spec^1(M', g')$ holds.

(i) For $m = 2, 3$ or $16 \leqq m \leqq 93$, (M, g) is of constant curvature c if and only if (M', g') is of constant curvature c.

(ii) For $m=4$, if M and M' have the same Euler-Poincaré characteristic $\chi(M) = \chi(M')$, then (M, g) is of constant curvature c if and only if (M', g') is of constant curvature c.

(iii) For $m = 3$ or $m = 15$, (M, g) is Einstein space if and only if $(M', g')$is an Einstein space.

In this paper we study the effect of $Spec^2(M, g) = Spec^2(M', g')$. For this we apply Patodi's result [4] on coefficients of Minakshisundaram-Pleijel-Gaffney's asymptotic expansion. The result obtained is following:

THEOREM: *Let (M, g) and (M', g') be compact connected orientable Riemannian manifolds. We assume that $Spec^2.(M, g) = Spec^2(M', g')$ holds good. Then*

(i) *for $m = 2, 3, 6, 7, 8, 14$, or $17 \sim 178$, (M, g) is of constant curvature K if and only if (M', g') is of constant curvature K,*

and

(ii) *for $m = 15$, or $m = 16$, (M, g) is an Einstein space if and only if (M', g') is an Einstein space.*

THEOREM says, in particular, that for $m = 2, 3, 6, 7, 8, 14, 17 \sim 178$, $Spec^2(M, g) = Spec^2(S^m, g_0)$ implies that (M, g) is isometric to an Euclidean sphere (S^m, g_0).

2. Proof of the theorem; By $R = (R^i{}_{jkl})$, $\rho = (\rho_{jk}) = (R^l{}_{jkl})$ and τ, we denote the Riemannian curvature tensor, the Ricci curvature tensor and the scalar curvature, respectively. We use the Minakshisundaram–Pleijel–Gaffney's asymptotic expansion which gives us the relation between the spectra of $\triangle$ and the curvature tensors of (M, g),

$$\sum_{k=1}^{\infty} exp\ (\lambda_{k,p}t) \underset{t \downarrow 0}{\sim} (4\pi t)^{-\frac{m}{2}} (a_{0,p} + a_{1,p}t + a_{2,p}t^2 + a_{3,p}t^3 + \cdots\cdots). \tag{2.1}$$

It may be noticed that instead of $Spec^2\ (M, g) = Spec^2\ (M'. g')$, we only use $a_{k,2} = a'_{k,2}$ for $k = 0, 1, 2$.
$a_{0,p}$, $a_{1,p}$ and $a_{2,p}$ in (2.1) were calculated by V. K. Patodi [4]. (cf. $a_{0,0}$, $a_{1,0}$ and $a_{2,0}$ were calculated by M. Berger [1] and H. P. Mckean–I. M. Singer [3]. $a_{3,0}$ was calculated by T. Sakai [5] and $a_{3,1}$ was by K.Ii [2].) We have

$$a_{0,2} = \binom{m}{2} \int_M dM\ , \tag{2.2}$$

$$a_{1,2} = \int_M \frac{m^2 - 13m + 24}{12} \tau dM \tag{2.3}$$

and

$$a_{2,2} = \int_M [C_1\ (m, 2)\ \tau^2 + C_2\ (m, 2)\ |\rho|^2 + C_3\ (m, 2)\ |R|^2]\ dM \tag{2.4}$$

where

$$C_1\ (m, 2) = \begin{cases} \dfrac{m^2 - 25m + 120}{144} & m \geqq 4, \\ -\dfrac{1}{8} & m = 3, \\ \dfrac{1}{72} & m = 2, \end{cases} \tag{2.5}$$

$$C_2\ (m, 2) = \begin{cases} \dfrac{-m^2 + 181m - 1080}{360} & m \geqq 4, \\ \dfrac{29}{60} & m = 3, \\ -\dfrac{1}{180} & m = 2, \end{cases} \tag{2.6}$$

and

$$C_3\ (m, 2) = \begin{cases} \dfrac{m^2 - 31m + 240}{360} & m \geqq 4, \\ -\dfrac{1}{15} & m = 3, \\ \dfrac{1}{180} & m = 2. \end{cases} \tag{2.7}$$

The Weyl's conformal curvature tensor $C = (C^i{}_{jkl})$, $C_{ijkl} = g_{ir}\ C^r{}_{jkl}$, is

$$(2.8)\qquad C_{ijkl} = R_{ijkl} - \frac{1}{m-2}(\rho_{jk}\, g_{il} - \rho_{jl}\, g_{ik} + g_{jk}\, \rho_{il} - g_{jl}\, \rho_{ik})$$

$$+ \frac{1}{(m-1)(m-2)}(g_{jk}\, g_{il} - g_{jl}\, g_{ik})\, \tau .$$

Now we put

$$(2.9)\qquad G = (G_{jk}) = (\rho_{jk} - \frac{1}{m}\tau g_{jk})$$

$$(2.10)\qquad G(g) = |G|^2 = |\rho|^2 - \frac{1}{m}\tau^2 ,$$

and

$$(2.11)\qquad C(g) = |C|^2 = |R|^2 - \frac{4}{m-2}|\rho|^2 + \frac{2}{(m-1)(m-2)}\tau^2 \quad m \geqq 3.$$

Then $G(g) \geqq 0$ holds; the equality holds on M if and only if (M, g) is an Einstein space. $C(g) \geqq 0$ holds also. By (2.10) and (2.11) $|R|^2$ and $|\rho|^2$ are written by $G(g)$, $C(g)$ and τ^2. Hence when $m \geqq 3$, from (2.4), (2.10) and (2.11), we get

$$(2.12)\qquad a_{2,2} = \int_M \Big\{\Big[C_1(m,2) + \frac{C_2(m,2)}{m} + \frac{2C_3(m,2)}{m(m-1)}\Big]\tau^2 + C_3(m,2)\, C(g)$$

$$+ \Big[C_2(m,2) + \frac{4C_3(m,2)}{m-2}\Big] G(g)\Big\}\, dM$$

We denote the coefficients of τ^2 and G (g) by $\Psi(m)$ and $\Phi(m)$ respectively, i. e.

$$(2.13)\qquad \Phi(m) = C_2(m,2) + \frac{4C_3(m,2)}{m-2} ,$$

$$(2.14)\qquad \Psi(m) = C_1(m,2) + \frac{C_2(m,2)}{m} + \frac{2C_3(m,2)}{m(m-1)} .$$

Then the following (2.15) and (2.16) are directly derived from (2.5), (2.6), (2.7), (2.13) and (2.14) .

$$(2.15)\qquad \begin{aligned} \Phi(m) &= -\frac{1}{360(m-2)}(m^3 - 187m^2 + 1566m - 3120) \\ & \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad m \geqq 4 . \\ \Psi(m) &= \frac{1}{720m(m-1)}(5m^4 - 132m^3 + 1093m^2 - 3246m + 3120) \end{aligned}$$

$$(2.16)\qquad \Phi(3) = \frac{13}{60} , \qquad \Psi(3) = \frac{1}{72}$$

From (2.1) ~ (2.4), the condition $Spec^2(M, g) = Spec^2(M', g')$ implies

$$(2.17) \quad \int_M dM = \int_{M'} dM' \quad ,$$

$$(2.18) \quad \int_M \tau dM = \int_{M'} \tau' dM \quad ,$$

and

$$(2.19) \quad \begin{aligned} &\int_M [C_3(m,2)\,C(g) + \Phi(m)\,G(g) + \Psi(m)\,\tau^2]\,dM \\ &= \int_{M'} [C_3(m,2)\,C(g') + \Phi(m)\,G(g') + \Psi(m)\,\tau'^2]\,dM' \end{aligned}$$

Now we assume furthermore that (M, g) is of constant curvature K. Then $C(g) = G(g) = 0$ hold on M and τ is constant on M. Therefore by (2.19), we get

$$(2.20) \quad \begin{aligned} &\int_{M'} [C_3(m,2)\,C(g') + \Phi(m)\,G(g')]\,dM' \\ &= \Psi(m)\,[\int_M \tau^2\,dM - \int_{M'} \tau'^2\,dM'] \end{aligned}$$

By using the Schwarz's inequality for τ', we get

$$(2.21) \quad \int_{M'} \tau'^2\,dM' \geqq \frac{[\int_{M'} \tau'\,dM']^2}{\int_{M'} dM'}$$

where the equality holds if and only if τ' is constant on M'. On the other hand, by (2.17), (2.18) and the fact that τ is constant on M, the right-hand side of (2.21) is $\int_M \tau^2\,dM$. So we get the inequality

$$(2.22) \quad \int_M \tau^2\,dM - \int_{M'} \tau'^2\,dM' \leqq 0$$

where the equality holds if and only if τ' is constant and equal to τ.

Therefore $C(g') = 0$, $G(g') = 0$ and $\tau' =$ constant $(= \tau)$ hold for m such that

$$(2.23) \quad m \geqq 3, \ C_3(m,2) > 0, \ \Phi(m) > 0 \text{ and } \Psi(m) \geqq 0.$$

We see easily that m which satisfies (2.23) runs 3, 6, 7, 8, 14, or 17～178.

It is known that $C(g') = 0$ and $G(g') = 0$ hold simultaneously if and only if (M', g') is of constant curvature. Hence (i) of the theorem is proved for the case $m = 3, 6, 7, 8, 14$, or 17～178.

When $m = 2$, (M, g) is automatically conformally flat, so we return to the original integral formula (2.4). Let K and K' be the Gaussian curvatures of (M, g) and (M', g') respectively. Then it is well-known that

$$(2.24)\qquad |R|^2 = 4K^2,\quad |\rho|^2 = 2K^2,\quad \tau^2 = 4K^2$$

hold good. And we get

$$(2.25)\qquad a_{2,2} = \int_M \frac{\tau^2}{60}\, dM \quad .$$

Therefore the condition $Spec^2\,(M, g) = Spec^2\,(M', g')$ implies

$$\int_M dM = \int_{M'} dM' \quad ,$$

$$(2.26)\qquad \int_M \tau dM = \int_{M'} \tau' \, dM' \ ,$$

$$\int_M \tau^2 dM = \int_{M'} \tau'^2 \, dM' \ .$$

If (M, g) is of constant curvature, then τ is constant. Similarly as in (2.21), by using the Schwarz's inequality, it is derived that τ' is also constant. This proves (i).

Finally we show (ii). When $m = 15$ or $m = 16$, $C_3\,(m,2) = 0$ holds good. Now we assume that (M, g) is an Einstein space, i.e. $G\,(g) = 0$. Then (2.19) implies

$$(2.27)\qquad \int_{M'} \Phi\,(m)\, G\,(g') d\,M' = \Psi\,(m)\, [\int_M \tau^2\, dM - \int_{M'} \tau'^2\, dM'] \quad .$$

We see that $\int_M \tau^2\, dM - \int_{M'} \tau'^2\, dM' \leqq 0$. The proof is similar to the above discussion. Therefore $\Phi(m) > 0$ and $\Psi(m) \geqq 0$ imply $G\,(g') = 0$, i.e. (M',g') is an Einstein space. This completes the proof of the theorem.

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