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COMPLETE ORIENTED SURFACES WITH NON-POSITIVE GAUSSIAN CURVATURE AND CONSTANT MEAN CURVATURE IN THE UNIT 3-SPHERE

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Abstract:

In this paper, we study the rigidity of surfaces in the unit 3-sphere with constant mean curvature. Under the condition that the surface is complete and has non-positive Gaussian curvature, we see that the surface must be flat and locally the standard product of circles.

§1. Introduction. H.B.Lawson, Jr. pointed out in his paper [7] that there is an explicit duality between minimal surfaces in the unit 3-sphere S^3 and surfaces of constant mean curvature 1 in the 3-dimensional Euclidean space E^3.

We can generalize this as follows : there is an explicit duality in the sense of H.B. Lawson, Jr. between surfaces of constant mean curvature H in S^3 and surfaces of constant mean curvature $\varepsilon(H^2+1)^{\frac{1}{2}}$ in E^3. In this paper we consider a certain property of surfaces of constant mean curvature in S^3 which is explicitly dual to that of surfaces of constant mean curvature in E^3.

We get the following theorem and corollary.

THEOREM: *Let (M^2, g) be an oriented, complete 2-dimensional Riemannian manifold with a Riemannian metric g and non-positive Gaussian curvature and x: $M^2 \longrightarrow S^3$ be an isometric immersion of M^2 into S^3 with constant mean curvature H.*

Then M^2 is flat and x: $M^2 \longrightarrow S^3$ is locally the standard immersion of product of circles z: $S^1(a_1) \times S^1(a_2) \longrightarrow S^3$,
where

$$a_1 = \frac{1}{2^{\frac{1}{2}}(H^2+1)^{\frac{1}{4}}[H+(H^2+1)^{\frac{1}{2}}]^{\frac{1}{2}}},$$

$$a_2 = \frac{[H+(H^2+1)^{\frac{1}{2}}]^{\frac{1}{2}}}{2^{\frac{1}{2}}(H^2+1)^{\frac{1}{4}}},$$

and $S^1(a_1)$, $S^1(a_2)$ mean circles with radii a_1 and a_2 respectively.

By the standard immersion we mean the immersion explained in S.S.Chern [2] *p.* 31–32.

COROLLARY: *In the above theorem, assume that M^2 is compact. Then up to rotations of S^3, $x : M^2 \longrightarrow S^3$ is the standard immersion of product of circles.*

REMARK 1. This corollary is not explicitly dual to the surfaces in E^3, because any compact surfaces with non-positive Gaussian curvature can not be immersed in E^3.

§2. Preliminaries.

Let M^2 be an oriented 2–dimensional Riemannian manifold and

$$(2.1) \qquad x: M^2 \longrightarrow S^3,$$

$$(2.1)' \qquad \tilde{x}: M^2 \longrightarrow E^3$$

be an isometric immersion of M^2 into S^3 (resp. E^3) .

Since M^2, S^3 and E^3 are oriented, we can choose globally the field ξ (resp. $\tilde{\xi}$) of the unit normal vectors of the image of the immersion (2.1) (resp. (2.1)′) such that $\{e_1, e_2, \xi\}$ (resp. $\{e_1, e_2, \tilde{\xi}\}$) belongs to the positive orientation of S^3 (resp. E^3) where $\{e_1, e_2\}$ is a positively oriented frame in M^2.

Let h (resp. $\tilde{h}$) be the second fundamental form of the immersion (2.1) (resp. (2.1)′) with respect to ξ (resp. $\tilde{\xi}$). Around any fixed point of M^2 we can choose local fields of positively oriented orthonormal vectors of M^2, say e_1, e_2. Let the components of h (resp. $\tilde{h}$) with respect to the frame $\{e_1, e_2\}$ be (h_{ij}) (resp. $(\tilde{h}_{ij})$), where i, j run through the range 1, 2. Then the Gauss, Codazzi equations of the immersion (2.1) (resp. (2.1)′) are

$$(2.2) \qquad K = 1 + (h_{11}h_{22} - h_{12}{}^2),$$

$$(2.2)' \qquad K = 0 + (\tilde{h}_{11}\tilde{h}_{22} - \tilde{h}_{12}{}^2),$$

$$(2.3) \qquad h_{ijk} = h_{ikj},$$

$$i,j,k = 1, 2$$

$$(2.3)' \qquad \tilde{h}_{ijk} = \tilde{h}_{ikj},$$

where K is the Gaussian curvature of M^2 and h_{ijk} (resp. $\tilde{h}_{ijk}$) is covariant derivative of h (resp. $\tilde{h}$) in the direction of e_k.

The mean curvature H (resp. $\tilde{H}$) of (2.1) (resp. (2.1)′) is given by definition

$$(2.4) \qquad H = \frac{1}{2} (h_{11} + h_{22}) ,$$

$$(2.4)' \qquad \tilde{H} = \frac{1}{2} (\tilde{h}_{11} + \tilde{h}_{22}) .$$

§3. Proof of the Theorem.

First we prove the following two lemmas which are direct extensions of those of H.B.Lawson, Jr. [7].

LEMMA 1. *Let $x: M^2 \longrightarrow S^3$ be an isometric immersion of constant mean curvature H and h be the second fundamental form of this immersion. Then*

$$(3.1) \qquad \tilde{h} = h + [\varepsilon(H^2+1)^{\frac{1}{2}} - H]\, g, \text{ where } \varepsilon = sgn\,(H) = \begin{cases} +1 & if\ H \geqq 0 \\ -1 & if\ H < 0 \end{cases}$$

is a symmetric (0, 2)-tensor field on M^2 which satisfies the Gauss, Codazzi relations for an isometric immersion of M^2 into E^3.

The trace of $\tilde{h}$ is $2\varepsilon(H^2+1)^{\frac{1}{2}}$.

PROOF. Since H is constant, the Codazzi relation (2.3)′ is obvious. It is enough to show the Gauss relation. To show it, let $\{e_1, e_2\}$ be a local field of orthonormal frames as in §2. With respect to these frames, (3.1) reduces to

$$\begin{cases} \tilde{h}_{11} = h_{11} + \varepsilon\,(H^2+1)^{\frac{1}{2}} - H, \\ \tilde{h}_{22} = h_{22} + \varepsilon\,(H^2+1)^{\frac{1}{2}} - H, \\ \tilde{h}_{12} = h_{12}, \end{cases}$$

and so

$$\begin{aligned} &\tilde{h}_{11}\tilde{h}_{22} - \tilde{h}_{12}{}^2 \\ &= h_{11}h_{22} - h_{12}{}^2 + [\varepsilon(H^2+1)^{\frac{1}{2}} - H]\,(h_{11}+h_{22}) + [\varepsilon(H^2+1)^{\frac{1}{2}} - H]^2 \\ &= K - 1 + 2H[\varepsilon(H^2+1)^{\frac{1}{2}} - H] + [\varepsilon(H^2+1)^{\frac{1}{2}} - H]^2 \\ &= K \end{aligned}$$

holds good, where we have used (2.2) and (2.4).

Q.E.D.

LEMMA 2. *Let $\tilde{x}: M^2 \longrightarrow E^3$ be an isometric immersion of constant mean curvature $\tilde{H}$ where $|\tilde{H}| \geqq 1$ and $\tilde{h}$ be the second fundamental form of this immersion. Then*

$$(3.2) \qquad \tilde{\tilde{h}} = \tilde{h} + [\tilde{\varepsilon}(\tilde{H}^2-1)^{\frac{1}{2}} - \tilde{H}]\, g, \text{ where } \tilde{\varepsilon} = sgn(\tilde{H}) = \begin{cases} +1 & if\ \tilde{H} \geqq 0 \\ -1 & if\ \tilde{H} < 0 \end{cases}$$

is a symmetric (0, 2)-tensor field on M^2 which satisfies the Gauss, Codazzi relations for an isometric immersion of M^2 into S^3.

The trace of $\tilde{\tilde{h}}$ is $2\tilde{\varepsilon}(\tilde{H}^2-1)^{\frac{1}{2}}$.

Since the proof of Lemma 2 is completely similar to that of Lemma 1, we may omit it.

When M^2 is simply-connected, Lemma 1, Lemma 2 and the fundamental theorem on surfaces in space forms (cf. for example S. Sasaki [9]) tell us that the following lemmas are true.

LEMMA 1′. *Let x: $M^2 \longrightarrow S^3$ be an isometric immersion with constant mean curvature H and h be the second fundamental form of this immersion. If M^2 is simply-connected, then there exists an isometric immersion $\tilde{x}$: $M^2 \longrightarrow E^3$ whose second fundametal form $\tilde{h}$ satisfies* (3.1).

LEMMA 2′. *Let $\tilde{x}$: $M^2 \longrightarrow E^3$ be an isometric immersion with constant mean curvature $\tilde{H}$ such that $|\tilde{H}| \geqq 1$ and $\tilde{h}$ be the seeond fundamental form of this immersion. If M^2 is simply-connected, then there exists an isometric immersion $\approx{x}$: $M^2 \longrightarrow S^3$ whose second fundamental form $\approx{h}$ satisfies* (3.2).

REMARK 2. Let Ω_{S^3} be a family of isometric immersions of the simply-connected surface M^2 into S^3 with constant mean curvature and Ω_{E^3} be a family of isometric immersions of M^2 into E^3 with constant mean curvature whose absolute value is no less than 1. Then LEMMA 1′ (resp. LEMMA 2′) induces a mapping Φ (resp. Ψ) of Ω_{S^3} (resp. Ω_{E^3}) into Ω_{E^3} (resp. Ω_{S^3}).

$$\Phi \circ \Psi = \text{identity},$$

and

$$\Psi \circ \Phi = \text{identity}$$

hold good.

Proof of the THEOREM: Let $\hat{M}^2$ be the universal covering manifold of M^2 with the natural Riemannian metric and π be the covering projection. It is known that if M^2 is complete, then $\hat{M}^2$ is also complete. Let $\hat{x}$ be the composition of π and x, i.e., $\hat{x} = x \circ \pi$. Then $\hat{x}$: $\hat{M}^2 \longrightarrow S^3$ is an isometric immersion and locally coinsides with x: $M^2 \longrightarrow S^3$. So the mean curvature of $\hat{x}$: $\hat{M}^2 \longrightarrow S^3$ is constant H and the Gaussian curvature $\hat{K}$ of $\hat{M}^2$ satisfies $\hat{K} = K \circ \pi \leqq 0$. By virtue of LEMMA 1′, there exists an isometric immersion $\tilde{\hat{x}}$: $\hat{M}^2 \longrightarrow E^3$ whose second fundamental form satisfies

$$\tilde{h} = \hat{h} + [\varepsilon (H^2 + 1)^{\frac{1}{2}} - H] \hat{g},$$

where $\hat{h}$ is the second fundamental form of $\hat{x}$: $\hat{M}^2 \longrightarrow S^3$ and $\hat{g}$ is the Riemannian metric of $\hat{M}^2$.

Since $\hat{K} \leqq 0$, $\hat{M}^2$ is complete and the mean curvature of $\tilde{\hat{x}}$: $\hat{M}^2 \longrightarrow E^3$ is non-zero constant $\varepsilon (H^2 + 1)^{\frac{1}{2}}$, we see that $\hat{K} = 0$ holds identically by a theorem of T. Klotz-R. Osserman [6] to the effect "*If $\tilde{x}$: $M^2 \longrightarrow E^3$ is an isometric immersion of non-zero* *constant mean curvature, M^2 is complete and the Gaussian curvature of M^2 is non-positive, then M^2 is flat*". Hence the Gaussian curvature of M^2 is identically zero.

Let k_1, k_2 be the principal curvatures of $x: M^2 \longrightarrow S^3$. Then the Gauss equation and the mean curvature H can be written as follows;

$$(3.3) \qquad K = 1 + k_1 k_2 = 0,$$

$$(3.4) \qquad k_1 + k_2 = 2H.$$

Therefore k_1 and k_2 are constant. On the other hand, the second fundamental form of the standard immersion $z: S^1(a_1) \times S^1(a_2) \longrightarrow S^3$ has the form

$$(3.5) \qquad \begin{pmatrix} \frac{a_2}{a_1} & 0 \\ 0 & -\frac{a_1}{a_2} \end{pmatrix} \qquad \text{where } a_1^2 + a_2^2 = 1.$$

We solve a_1, a_2 under the conditions

$$(3.6) \qquad a_1^2 + a_2^2 = 1,$$

$$(3.7) \qquad \frac{a_2}{a_1} - \frac{a_1}{a_2} = 2H$$

and we get

$$(3.8) \qquad a_1 = \frac{1}{2^{\frac{1}{2}} (H^2+1)^{\frac{1}{4}} [H + (H^2+1)^{\frac{1}{2}}]^{\frac{1}{2}}}, \qquad a_2 = \frac{[H + (H^2+1)^{\frac{1}{2}}]^{\frac{1}{2}}}{2^{\frac{1}{2}} (H^2+1)^{\frac{1}{4}}}.$$

When a_1, a_2 are as in (3.8), the second fundamental form of

$$x: M^2 \longrightarrow S^3$$

and

$$z: S^1(a_1) \times S^1(a_2) \longrightarrow S^3$$

coinside locally. Hence they are locally isometric to each other by the fundamental theorem on surfaces in S^3. This completes the proof of the theorem. Q.E.D.

§4. **Remarks.** Lemma 1′ seems to derive directly the following famous result (ii) and recent work (iii) from (i).
(i) and (ii), (iii) are explicitly dual to each other in the sense of H. B. Lawson, Jr. [7] by virtue of Lemma 1′ and Lemma 2′.

(i) Let $\tilde{x}: M^2 \longrightarrow E^3$ be an isometric immersion with non-zero constant mean curvature. If M^2 is homeomorphic to S^2, then $\tilde{x}: M^2 \longrightarrow E^3$ is umbilical and $\tilde{x}(M^2)$ is a 2-sphere in E^3. (H.Hopf [3])

(ii) Let $x: M^2 \longrightarrow S^3$ be a minimal immersion. If M^2 is homeomorphic to S^2, then $x: M^2 \longrightarrow S^3$ is totally geodesic and $x(M^2)$ is a great 2-sphere in S^3. (F. J. Almgren [1])

(iii) Let $x: M^2 \longrightarrow S^3$ be an isometric immersion with constant mean curvature. If M^2 is homeomorphic to S^2, then $x: M^2 \longrightarrow S^3$ is umbilical and $x(M^2)$ is a great or a small 2-aphere in S^3. (K. Kenmotsu [4])

T. Klotz-R. Osserman[6] studied also the case where K is non-negative;

(iv) Let $\tilde{x}: M^2 \longrightarrow E^3$ be an isometric immersion with constant mean curvature. If M^2 is complete and has non-negative Gaussian curvature, then $\tilde{x}: M^2 \longrightarrow E^3$ is a 2-sphere, a plane or a right circular cylinder in E^3. (T. Klotz-R. Osserman [6])

The partial extensions and an explicitly dual expression of (iv) were given by K. Nomizu-B. Smyth [8]. We can give another proof for the dual of (iv).

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The paper above involves my result. But the methods of proof are independent and the proof of this paper is more simple and more elementary than that of the preceeding paper.

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