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Multifractal Analysis for Quantifying the Morphology of the Cluster of Galaxies in N-body Simulations

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ABSTRACT—Multifractal analysis for objectively examining the morphology of the cluster of galaxies is described. This analysis is applied to cosmological N-body simulations with power-law spectra. From this analysis, we find that the generalized dimension is useful statistics for quantifying the morphological difference of the clusters, which evolve from different initial conditions.

1 Introduction

It is known that the analysis of the large scale structure of the universe is a powerful method to decide the cosmological parameter of our universe. For this reason, many efforts have been done to quantify the galaxy distribution in an objective manner. Correlation function (Totsuji and Kihara 1969) is the most famous statistical measure used for the galaxy clustering. From the analysis of the two-point correlation function, the clustering properties of galaxies and clusters are resemble, although the scales are very different. More precisely the two-point correlation functions of galaxies and clusters follow same scaling property,

$$\xi(r) = \left(\frac{r}{r_0}\right)^{\gamma}, \tag{1}$$

where $\gamma = -1.8$ (Peebles 1980 ; Davis and Peebles 1983 ; Bahcall and Soneira 1983 ; Klypin and Kopylov 1983 ; Postman, Geller and Huchra 1988 ; Olivier et al. 1990).

This property was considered the evidence of the fractal nature of the galaxy distribution (Pietronero 1987 ; Coleman, Pietronero and Sanders 1988). However, purely fractal pattern of the galaxy distribution conflicts with the Cosmological Principal. Jones et al. (1988) and Martinez et al. (1990) showed that the galaxy distribution cannot be described by a pure fractal. In nonlinear regime, galaxy distribution is self-similarity, while a transition toward homogeneity is observed in linear regime. Furthermore, even in nonlinear regime, galaxy distribution cannot be described by a simple-fractal, but by a multifractal. By using cosmological N-body simulations, Valdarnini et al. (1992) showed that the multifractal structure of galaxies can naturally arise in the framework of the gravitational instability picture. In addition to the galaxy distribution, multifractal analysis of cluster distribution was done by Borgani et al. (1993).

We here apply multifractal analysis to the morphology of the cluster of galaxies in N-body

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simulations. It is apparent to the eye that the morphologies of the clusters of galaxies, which evolve from different initial conditions, are not same. Therefore, in addition to the analysis of the large scale structure, morphological analysis is also important for the cosmological probes. However the question is whether multifractal analysis is suitable to quantify the morphology of the cluster of galaxies. In this paper, we examine the usefulness of this analysis by using the cosmological N-body simulations.

The rest of the paper is organized as follows : The general survey of fractal and multifractal are in section 2. The simulation data and cluster finding algorithm are described in section 3, and in section 4 the results of our analysis are explained. Finally we discuss conclusions in section 5.

2 Summary of Multifractal Analysis

2. 1 Fractal and Multifractal

We consider for any $\varepsilon > 0$ the set of all the possible covering of a given set E, having diameters $\varepsilon_i \leq \varepsilon$. We introduce $M_D(E)$ as

$$M_D(E) = \lim_{\varepsilon \to 0} \inf_{\varepsilon_i < \varepsilon} \sum_i \varepsilon_i^D . \tag{2}$$

This expression defines the Hausdorff dimension D_H as the unique value of D that renders finite $M_D(E)$, with it vanishing for $D > D_H$ and diverging for $D < D_H$. A definition of a fractal set is a mathematical object whose Hausdorff dimension D_H is strictly larger than its topological dimension D_T (Mandelbrot 1982). Because we regard galaxy as point-like object, topological dimension is $D_T = 0$. Therefore if the Hausdorff dimension of a cluster is nonzero, cluster is a fractal set.

Unfortunately Hausdorff dimension only expresses a simple-fractal morphology. In order to describe more complicated morphology, we have to introduce a generalized dimension D_q. We divide a simulation cube into cells with side r. The total number of cells which cover this cube completely is represented by $N_{cell}(r)$. Then, the generalized dimension D_q (Renyi dimension) is defined as

$$D_q \equiv \frac{\tau(q)}{q-1}, \tag{3}$$

where q is an arbitrary real number. $\tau(q)$ is

$$\tau(q) = \lim_{r \to 0} \frac{\log \sum_{i=1}^{N_{cell}} [p_i(r)]^q}{\log r}, \tag{4}$$

with a cell occupancy probability of $p_i = N_i(r)/N$, where $N_i(r)$ is the number of particles in the i-th cell and $N \equiv \sum N_i$ is the total number of particles (Renyi 1970).

If we set $q = 0$, we obtain a capacity dimension D_0. Strictly speaking, capacity dimension is not equivalent to Hausdorff dimension. In the practical case, however, it is often considered that these are completely equivalent. In this paper, we also follow this.

2. 2 Correlation Integral Method

Renyi's method is the most famous definition of a generalized dimension. However, it is known that the difficulty of taking the $r \to 0$ limit properly in $\tau(q)$ exists. In fact, this definition dose not work well when we apply it to the galaxy distribution map (Ueda et al. 1993). In order to avoid this difficulty, we must estimate $\tau(q)$ at finite r, instead of taking the $r \to 0$ limit. Here we adopt correlation integral method which satisfies the above requirement (Valdarnini et al. 1992 ; Borgani et al. 1993 ; Ueda 1995).

The correlation integral method was proposed by Grassberger and Procaccia (1983). In this approach the partition function is defined

$$Z(q, r) = \frac{1}{N} \sum_{i=1}^{N} C_i(r)^{q-1}, \tag{5}$$

where $C_i(r)$ is an occupancy probability defined as $C_i(r) = N_i(\leq r)/N$, with $N_i(\leq r)$ being the number of particles in a sphere with radius r centered on the i-th object. If partition function satisfies the power-law relation, we can determine $\tau(q)$ from $Z(q, r) \propto r^{\tau(q)}$. Then the generalized dimension can be obtained from equation (3).

It, is known that correlation integral method with positive q works well when we apply this to the cosmological N-body simulation. In negative q, this analysis have a discreteness effect. Fortunately, we are interested in the morphology of cluster which corresponds to a positive q. (In $q > 0$ case, p_i with overdense region is weighted, while underdense region is weighted in $q < 0$ case.) Therefore, we do not suffer above problem. We also point out that D_q in correlation integral method is known as a close approximation of Renyi dimension. We therefore regard D_q which derives from this method as the equivalent form of Renyi dimension.

3 Models and Cluster-finding Algorithm

3. 1 Models

In order to examine the usefulness of multifractal analysis, we use the cosmological N-body simulations (Suginohara et al. 1991). We adopt, four simulations with different power-law initial conditions

$$P(k) \propto k^n \quad (n = 1, 0, -1, -2), \tag{6}$$

where $P(k)$ is the power spectrum of the initial density fluctuations. All of the simulations employ $N = 262{,}144$ particles, and are carried out in a cubic volume of L_b^3 with a periodic boundary condition (L_b is a cubic length). Hereafter we call these four models $n = 1$, $n = 0$, $n = -1$, $n = -2$ respectively. The above four models are evolved in an Einstein-de Sitter universe with density parameter $\Omega = 1.0$. The gravitational softening length is $L_b/1280$. Particle distributions of these models are at $a/a_i = 18.2$ ($n = 1$), 37.6 ($n = 0$), 6.9 ($n = -1$) and 6.7 ($n = -2$), where a is a cosmic scale factor and a_i denotes its initial value. After this, we normalize $L_b = a_i = 1$.

3. 2 Cluster-finding Algorithm

The idntification of *clusters* of *galaxies* from the simulation data is somewhat ambiguous, but essential in the present study. Here we adopted an adaptive linking method (ALM) developed by Suto et al. (1992). ALM refines the more conventional Friends-of-Friends algorithm which assumes the constant linking length. It uses the variable linking length b_{ij}

between i-th and j-th galaxies depending on the local density. More precisely, b_{ij} is defined as

$$b_{ij} = \mathrm{Min}\left[\frac{1}{N^{1/3}}, \frac{\beta(\rho_i(r_s)^{-1/3} + \rho_j(r_s)^{-1/3})}{2}\right] \tag{7}$$

where

$$\rho_i(r_s) = \frac{1}{(2\pi r_s^2)^{3/2}} \sum_{j=1}^{N} \exp\left(-\frac{|\mathbf{r}_i - \mathbf{r}_j|}{2 r_s^2}\right) \tag{8}$$

is a local density of i-th galaxy and $\mathbf{r}_i$ is a position vector of the i-th galaxy. Thus ALM requires two parameters, the proportional constant β and the smoothing length r_s. After the particles are properly grouped, we removed particles which are not gravitationally bound.

We adopt $(r_s, \beta) = (1/64, 0.4)$, and find clumps of particles in N-body simulation (see Ueda et al. 1993). The results are in figure 1. In this figure, we show the x-y projection of three clumps in each simlation. From these panels, it is clear that the morphologies of the clumps, which evolve from different initial conditions, are not same. Here, we define a *cluster of galaxies* as a clump which contains more than 200 particles.

4 Analysis

In this section, we explain the results of the multifractal analysis. Especially we pay attention to the generalized dimension, and examine whether this is useful to detect the difference of morphologies between clusters, which evolve from different initial conditions.

In order to estimate the generalized dimension, we have to calculate the partition function. For this purpose, we apply correlation integral method (equation (5)) to each cluster in figure 1 and obtain $Z(q, r)$ with order $q = 0, 3$. The results are in upper panels of figure 2a, 2b, 2c, 2d. In these figures, open triangles, squares and circles correspond to the $Z(q, r)$ of a cluster in left, middle and right panels in figure 1. In lower panels, we also plot the generalized dimension D_0, D_3 as a function of r. Notice that the generalized dimension is obtained by realizing a five-point local linear regression on the partition function (Borgani et al. 1993).

From lower panels in figure 2, the generalized dimension depends on the scale r. In order to see this feature minutely, we first pay attention to D_0. As stated in previous section, D_0 corresponds to Hausdorff dimension. In large r region, a cluster is recognized a small clump. In small r, we estimate a dimension of each particle in a cluster. Then in large and small scales, the dimension approaches the value of $D_0 \sim 0$. Only in medium scale, we recognize the characteristic dimension of a cluster. This simple supposition explains this behavior in figure 2. D_3, on the other hand, dose not correspond to Hausdorff dimension. Therefore, we cannot explain the behavior of D_3 in a simple manner.

Figure 2 suggests that the morphology of the cluster is very complicated. This is because that D_q has no flat shape, and we cannot determine D_q as a constant value. From these panels, it is soon noticed that the morphological structures of the clusters in each simulation are similar. In $q = 0$ case, the generalized dimension in $r \geq 4 \times 10^{-3}$ region dose not depend on each cluster. On the other hand, D_0 depends on each cluster in $r \leq 4 \times 10^{-3}$ region. Therefore, we consider that the clusters which evolve from same initial condition, have same morphology as far as $r \geq 4 \times 10^{-3}$ region. In $q = 3$ case, the generalized dimension does not depend on each

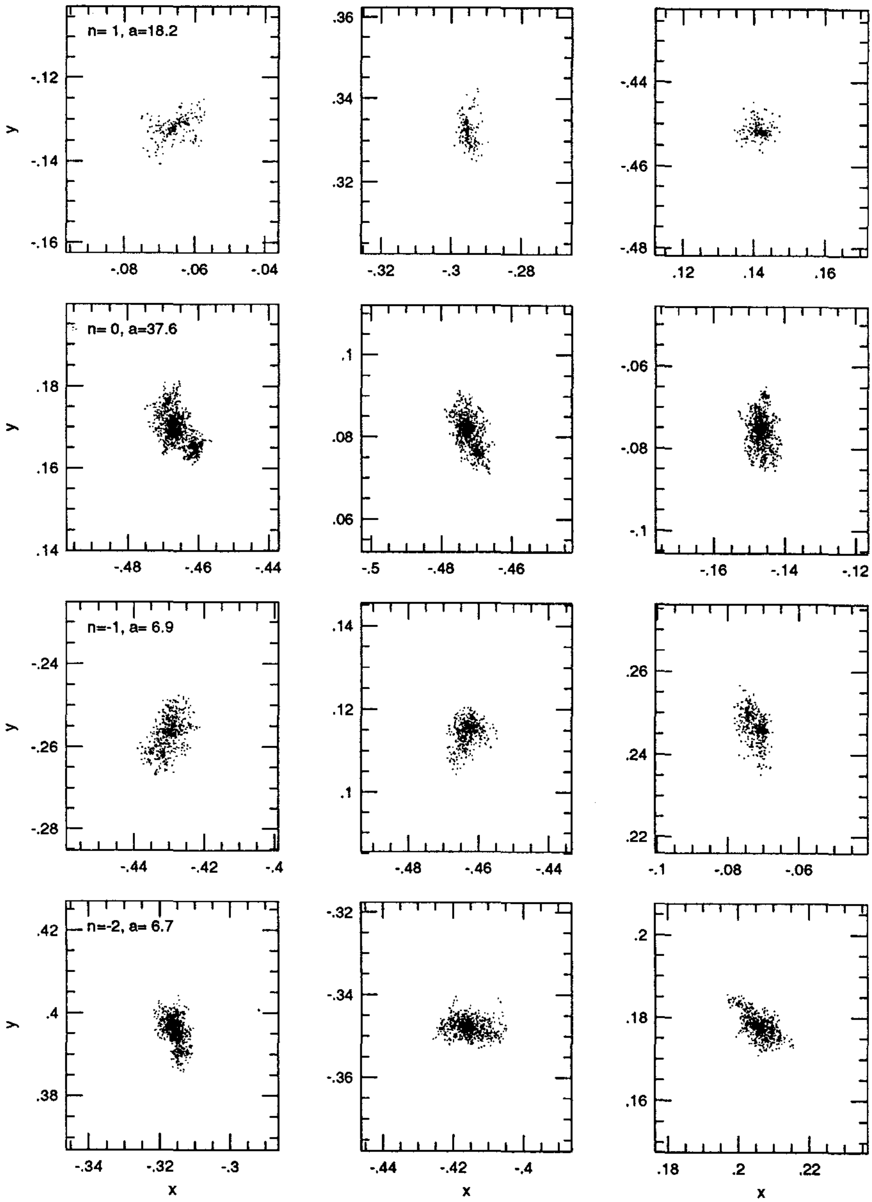

Figure 1. *x-y* projection of the clumps of particles in N-body simulations.

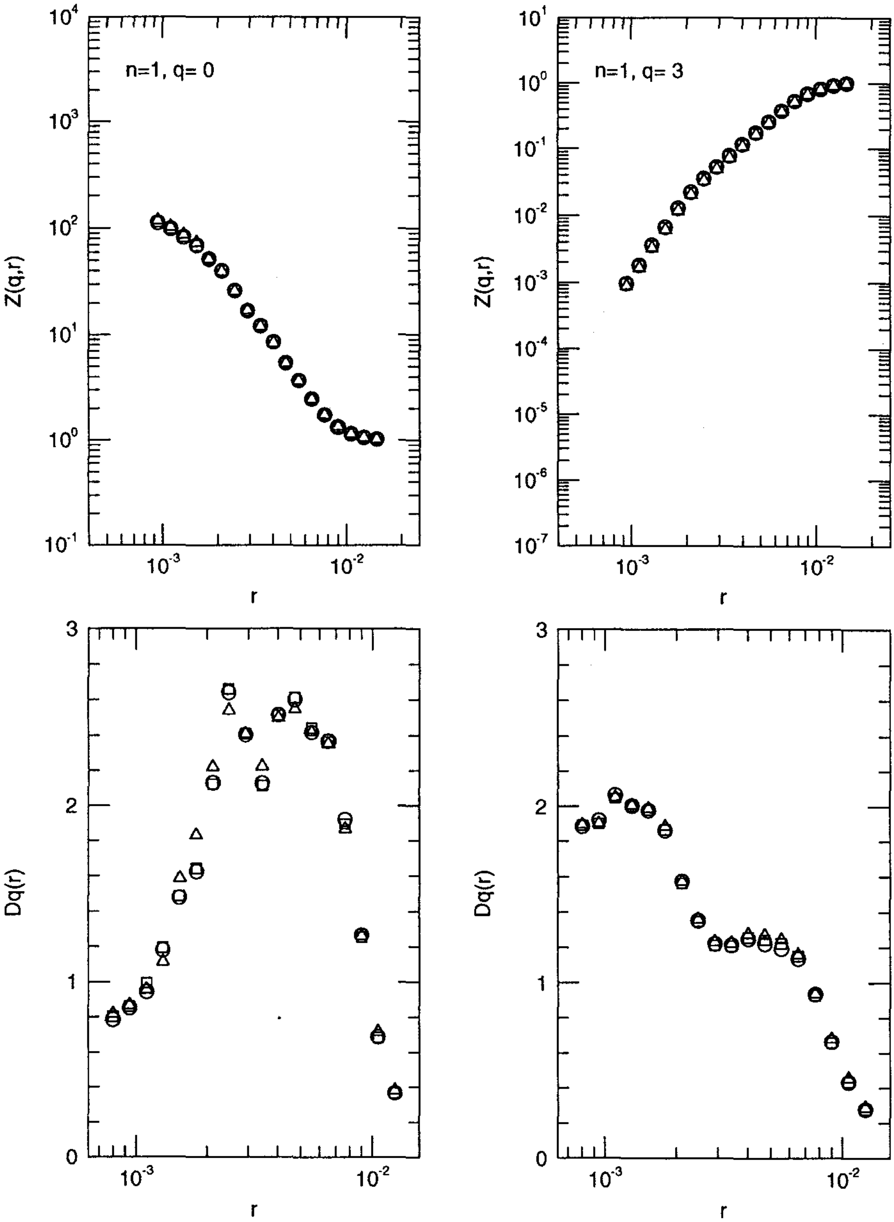

Figure 2a. Distribution function and the generalized dimension of $n = 1$ model.

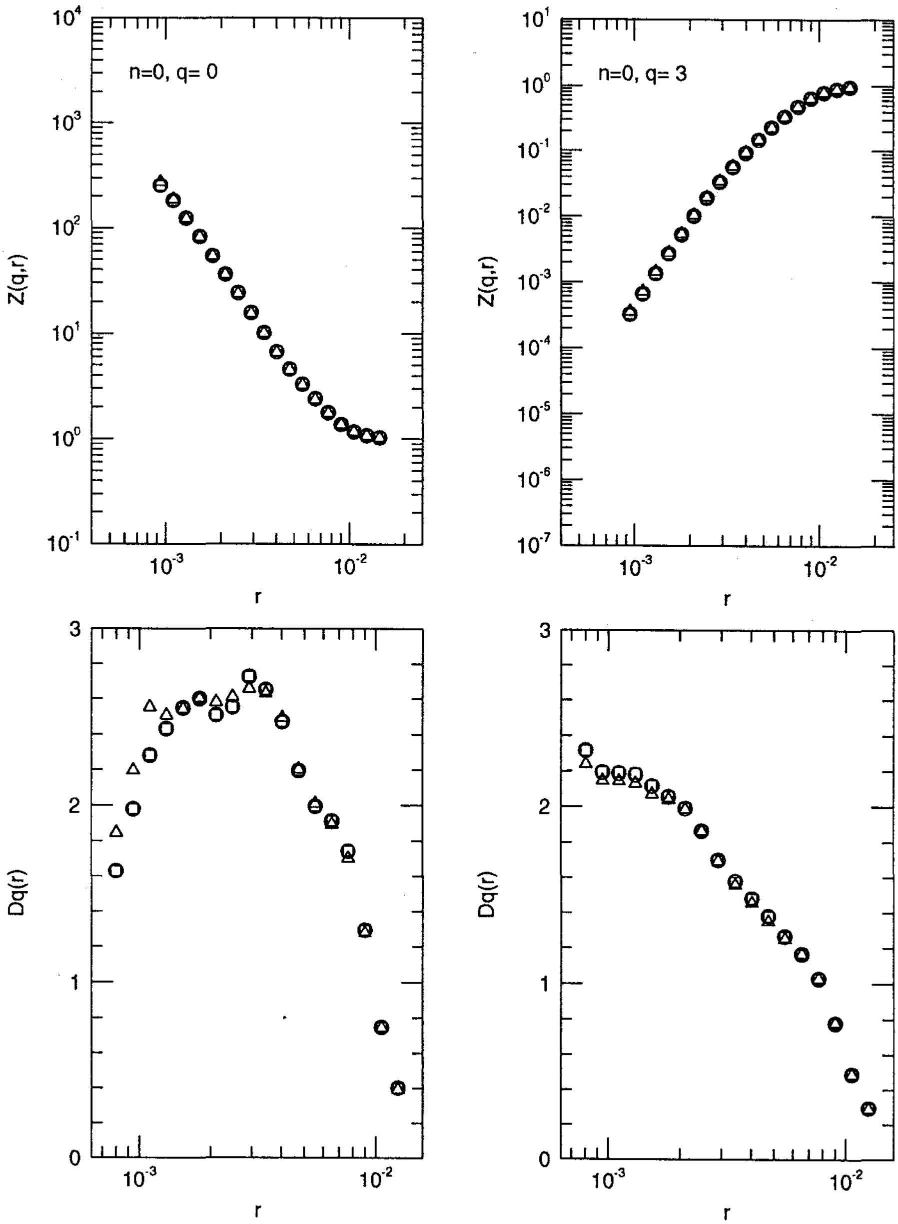

Figure 2b. Distribution function and the generalized dimension of $n = 0$ model.

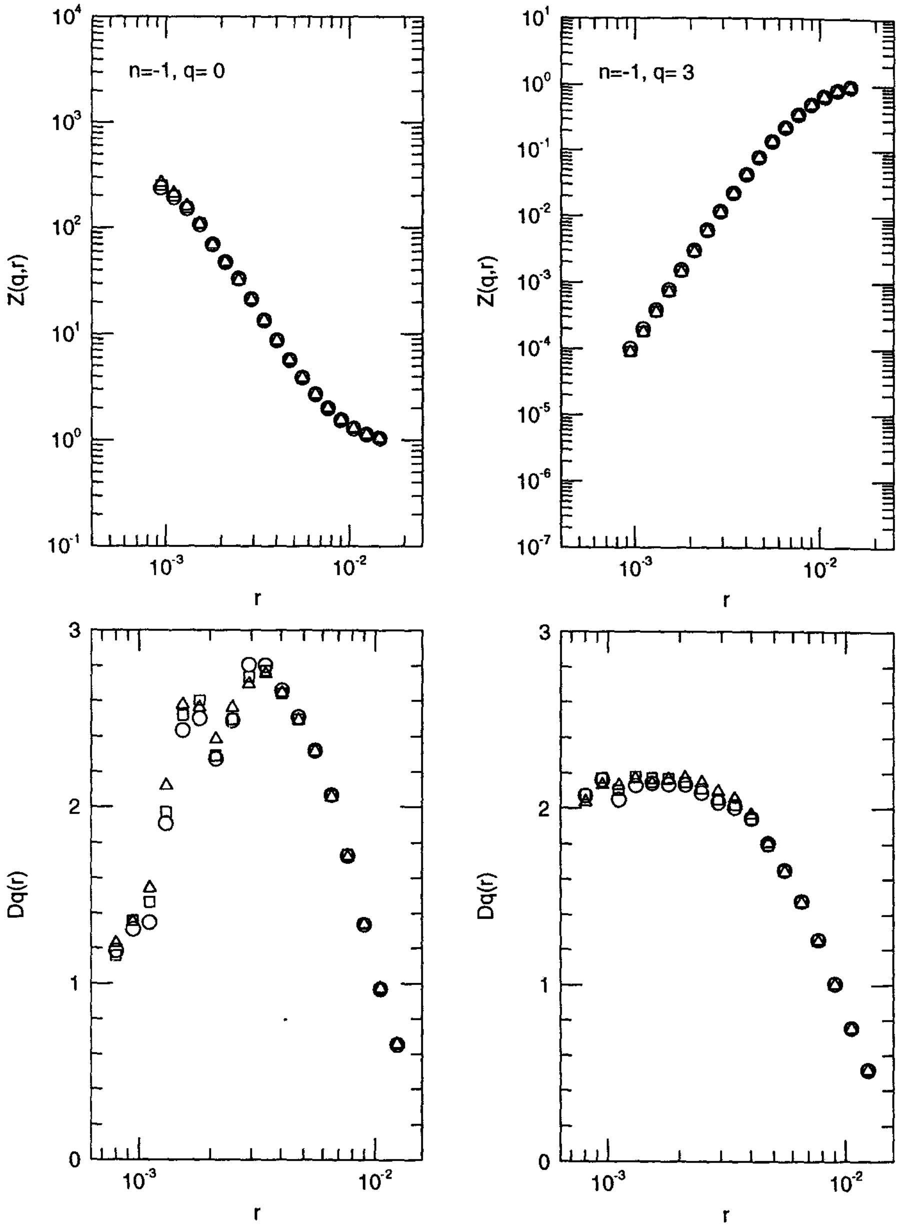

Figure 2c. Distribution function and the generalized dimension of $n = -1$ model.

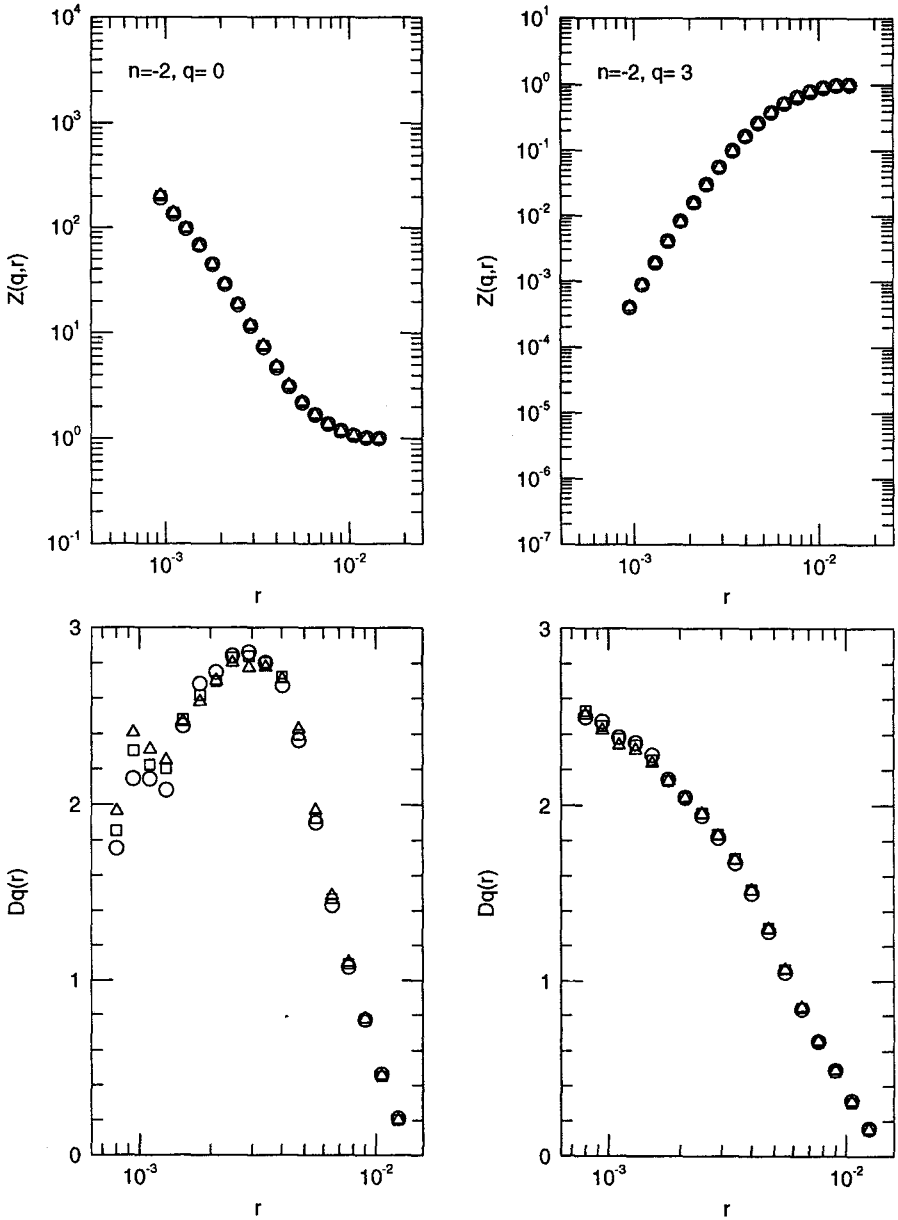

Figure 2d. Distribution function and the generalized dimension of $n = -2$ model.

cluster in $r \geq 1 \times 10^{-3}$ region. Then we conclude that the generalized dimension D_0, D_3 have an information about initial condition of a simulation in large r region. In addition to D_0 and D_3, we also analyze D_4, D_5. The results, however, are same in D_3 case.

In order to see the model dependence of the morphology of clusters, we compare the distribution function and the generalized dimension of clusters in $n=1$ 0, -1, -2 models.

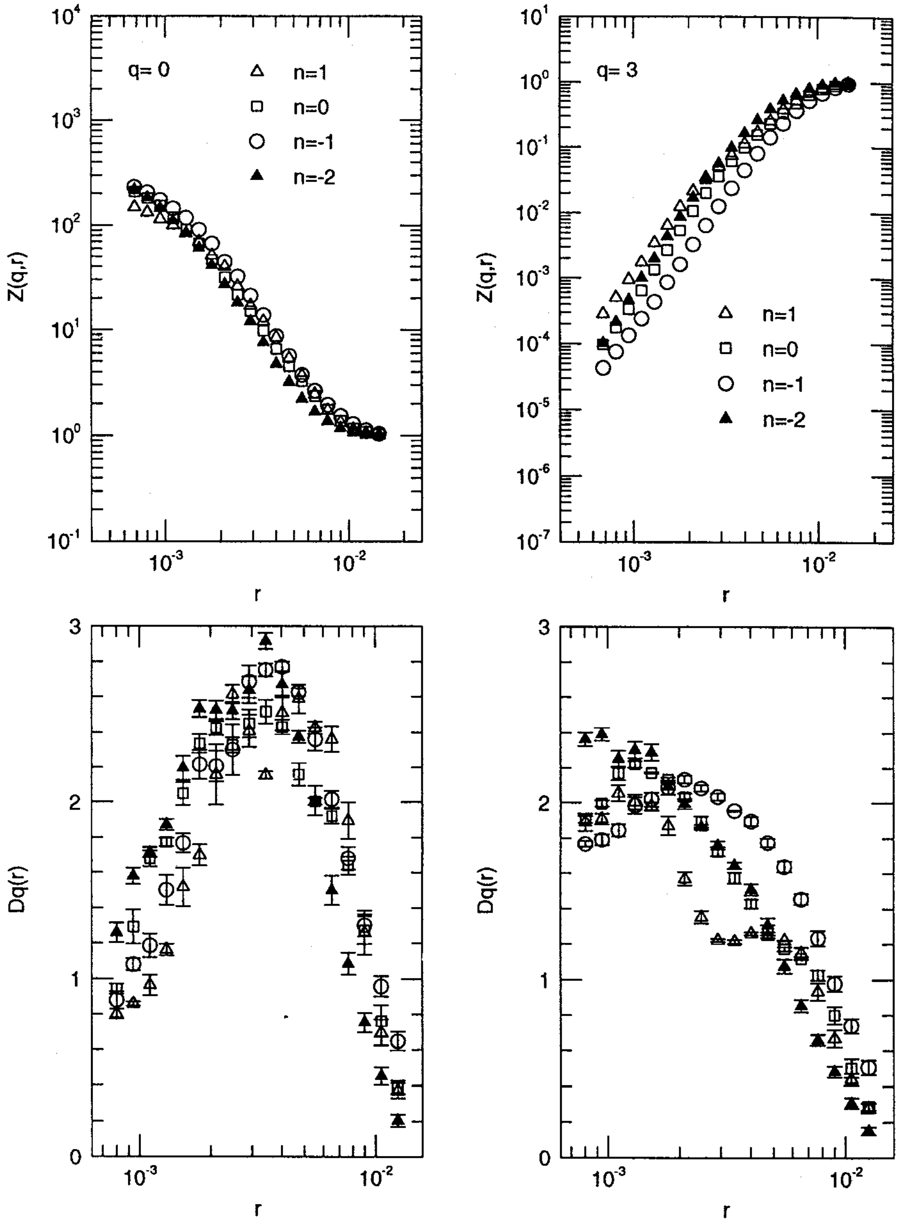

Figure 3. Averaged distribution function and the generalized dimension of $n=1$ (otriangle), $n=0$ (open aquare), $n=-1$ (open circle), $n=-2$ (triangle) models.

These results are in figure 3. To improve statistical reliability, we average $Z(q, r)$ of clusters in each simulation. From this average distribution function, we also estimate the generalized dimension D_0, D_3. To represent this average effect, we estimate the error in lower panels in this figure. From upper panels, one soon notice that the $Z(q, r)$ of four models are different with each other. So one may consider that the partition function is useful statistics to detect the difference of the morphology of clusters between models. From equation (5), however, the partition function depends on the number of galaxies in each cluster. Therefore, the difference of partition function is appearance. The generalizecl dimension D_q, on the other hand, dose not depend on the galaxy number in a cluster. Therefore this statistical measure is favorable.

It is very important to notice the fact that we can discriminate the morphology of clusters in four models by means of the generalized dimension. We first consider D_0 case. In $r \geq 4 \times 10^{-3}$, D_0 of $n = -2$ model is smaller than other models. On the other hand, D_0 of $n = -1$ model is the largest in $r \geq 1 \times 10^{-2}$. In $4 \times 10^{-3} \leq r \leq 8 \times 10^{-3}$, D_0 of $n = 1$ model is the largest.

We also consicler D_3 case. Notice that the error bar in D_3 is smaller than that in D_0. Therefore, we can discriminate four models more clearly by using D_3. In $r \geq 1 \times 10^{-3}$, D_3 of these four models are clearly different. (Although D_3 of $n = 0$ and $n = -2$ models are almost same in $1 \times 10^{-3} \leq r \leq 5 \times 10^{-3}$, the difference will appear in $r \geq 5 \times 10^{-3}$. And D_3 of $n = 1$ and $n = 0$ models are almost same in $r \geq 5 \times 10^{-3}$, but the difference will appear in $1 \times 10^{-3} \leq r \leq 5 \times 10^{-3}$.) Then we conclude that the generalized dimension at large scale region has informations about initial condition of a simulation.

5 Conclusions

In this paper, we apply multifractal analysis to the morphology of the cluster of galaxies in N-body simulations. The morphologies of the clusters, which evolve from different initial conditions, are not same. Therefore objective description of cluster morphology is an important for the cosmological probes. We examine the generalized dimension D_q with order $q = 0, 3$. From our analysis, the generalized dimension of clusters depend strongly on the scale, so we cannot determine this as a constant value. In large scale, however, we found that the generalized dimensions of the clusters in a simulation have same value. So, it is considered that the generalized dimension has informations about initial condition of a simulation. We compared the generalized dimension of the clusters between four models, and found that the behaviors of D_3 are very different with each other. Therefore, we conclude that the generalized dimension is useful statistics for quantifying the morphological difference of the clusters of galaxies, which evolve from different initial conditions.

Finally we mention the remaining problems of multifractal analysis. We apply multifractal analysis to power-law cosmological N-body simulations, so it is interesting to apply this analysis to cold dark matter models or observational results. And further analysis about cold dark matter models will be examined.

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