Scaling Exponents of Self-Similar Functions and Wavelet Analysis

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Abstract

In this paper we give estimations of the pointwise scaling exponents of self-similar functions on the $n$ dimensional Euclidean space $\mathbb{R}^n$. These estimations are derived by using a technique based on wavelet analysis. Examples of such self-similar functions include indefinite integrals of self-similar measures on $\mathbb{R}$ and they also include widely oscillatory functions (e.g. the Takagi function, the Weierstrass function and Lévy’s function). Pointwise scaling exponents provide an objective description of an irregularity of a function at a point. Our results are applied to compute the scaling exponents of several oscillatory functions.

1 Introduction.

Recently fractal sets and extremely irregular functions are playing an important role in physics, in image or signal processing and in mathematics (e.g. see [6]).

A pointwise scaling exponent of an irregular function $f(x)$ at a point $x_0$ is aimed to provide an objective description of the irregularity of $f(x)$ at $x_0$. There are several non-equivalent definitions of scaling exponents. In particular two scalings of the Hölder scaling exponent $\alpha(f, x_0)$ and the weak scaling exponent $\beta(f, x_0)$ have been investigated in Y. Meyer [7] whose works are based on relation between scaling exponents and estimating the size of wavelet transforms. The weak scaling exponent $\beta(f, x_0)$ is more sensitive to the oscillations of $f(x)$ at $x_0$. If $\alpha(f, x_0) = \beta(f, x_0)$, then $x_0$ is called a cusp singularity for a function $f(x)$. If $\alpha(f, x_0) \neq \beta(f, x_0)$, then $x_0$ is called an oscillating singularity for $f(x)$. Oscillating behavior of a function $f(x)$ at a point $x_0$ is relevant to the two scaling exponents of $f(x)$ at $x_0$. The pointwise Hölder scaling exponent of self-similar functions have been studied in relating to multifractal formalism in some particular case in [1], [2], [4] and [5].

In this paper we give the estimations of the two pointwise scaling exponents of $\alpha(F, x_0)$ and $\beta(F, x_0)$ for self-similar functions $F(x)$ on $\mathbb{R}^n$ in a more general setting. Examples of such self-similar functions include indefinite integrals of self-similar measures on $\mathbb{R}$ and they also include widely oscillatory functions (e.g. the Takagi function, the Weierstrass function and Lévy’s function). Properties of self-similar functions are closely...
related to ones of self-similar measures on $\mathbb{R}^n$. Basic properties of finite positive self-similar measures on $\mathbb{R}^n$ have been investigated systematically by S. Strichartz [8], [9] and [10] based on Fourier analysis.

In the paper, our results can be applied to compute the pointwise scaling exponents of oscillating functions in the examples.

The plan of next sections in our paper is as follows:

In the second section the several scaling exponents of a function are defined and we derive their basic properties.

In the third section we give the definition of self-similar functions and we prove the main theorem for scaling exponents of self-similar functions. In the proof of the theorem, a method of wavelet analysis is used.

In the fourth section we give examples of self-similar functions and we compute the scaling exponents of these examples by applying the theorem.

We use $C$ to denote a positive constant different in each occasion. But it will depend on the parameter appearing in each problem. The same notations $C$ are not necessarily the same on any two occurrences.

## 2 Scaling exponents.

If $\mathcal{S}(\mathbb{R}^n)$ denotes the Schwartz space, then $\mathcal{S}_0(\mathbb{R}^n)$ is the closed subspace of $\mathcal{S}(\mathbb{R}^n)$ defined by

$$\int y^\alpha \varphi(y)dy = 0, \quad \forall \alpha \in \mathbb{Z}_+^n$$

where $\mathbb{Z}_+$ is the set of all nonnegative integers. Let $F$ be a tempered distribution and $s$ a nonnegative real number. We write $F \in \Gamma^s(x)$ if for every $\varphi$ in $\mathcal{S}_0(\mathbb{R}^n)$, there exists a constant $C$ such that

$$|t^{-n} \int \varphi \left( \frac{y-x}{t} \right) F(y)dy| \leq Ct^s, \quad 0 < t \leq 1.$$ 

**Lemma 1** (c.f. [7 : Theorem 3.4]).

Let us consider an integer $k$ larger than a nonnegative number $s$ and a function $\varphi \in C^\infty(\mathbb{R}^n)$ with compact support such that

$$\int y^\alpha \varphi(y)dy = 0, \quad \forall |\alpha| \leq k.$$ 

Then if $F \in \Gamma^s(x_0)$, there exist positive constants $C$, $\delta_0$ for any given $C_0 > 0$ such that

$$|t^{-n} \int \varphi \left( \frac{R(y-x)}{t} \right) F(y)dy| \leq Ct^s$$

for any isometry $R$ whenever $|x-x_0| \leq C_0 t$ and $0 < t \leq \delta_0 < 1$.

A pointwise weak scaling exponent $\beta(f, x)$ for $x \in \mathbb{R}^n$ is defined as

$$\beta(f, x) = \sup\{s \geq 0 : f \in \Gamma^s(x)\}. \quad (1)$$
If \( k < s < k + 1 \) for a nonnegative integer \( k \), then for \( x \in \mathbb{R}^n \), a function \( f \in C^s(x) \) means that there exits a polynomial \( P_x \) of degree less than or equal to \( k \) such that

\[
|f(y) - P_x f(x-y)| \leq C |x-y|^s
\]  

(2)
on a neighborhood of \( x \). For an open set \( \Omega \) in \( \mathbb{R}^n \), \( f \in C^s(\Omega) \) means that \( f \) is bounded on \( \Omega \) and (2) holds for all \( x \in \Omega \) with a uniform constant \( C \) on \( \Omega \). When \( s \) is a nonnegative integer, we need some modification for the definition above. See for details [7: p.6].

The pointwise Hölder scaling exponent of a function \( f \) at a point \( x \) is defined as

\[
\alpha(f, x) = \sup\{s \geq 0 : f \in C^s(x)\}.
\]  

(3)
We also define

\[
\alpha(f, \Omega) = \sup\{s \geq 0 : f \in C^s(\Omega)\}.
\]  

(4)
In particular when \( \Omega = \mathbb{R}^n \), we write

\[
\alpha(f) = \alpha(f, \mathbb{R}^n).
\]  

(5)

Let \( B^s_{pq}(\mathbb{R}^n) \) and \( F^s_{pq}(\mathbb{R}^n) \) be the Besov space and the Triebel-Lizorkin space respectively for \( 0 < p, q \leq \infty \). Then we define

\[
\alpha_{pq}(f) = \sup\{s \geq 0 : f \in B^s_{pq}(\mathbb{R}^n)\},
\]

\[
\alpha_p(f) = \alpha_{p\infty}(f).
\]

By the definition it follows that \( \alpha(f) = \alpha_{\infty}(f) \) where \( \alpha(f) \) is given in (5). By the embedding theorem of the function space theory (e.g. see [11]), we have a following proposition :

**Proposition 1**

(a) \( \alpha_{pq}(f) = \sup\{s \geq 0 : f \in F^s_{pq}(\mathbb{R}^n)\} \), for \( 0 < p < \infty, \ 0 < q \leq \infty \).

(b) \( \alpha_p(f) = \alpha_{pq}(f) \) for \( 0 < p, \eta \leq \infty \).

(c) \( \alpha(f) \geq \alpha_p(f) - \frac{n}{p} \geq \alpha_q(f) - \frac{n}{q} \) for \( 0 < q \leq p < \infty \).

Let for \( x \in \mathbb{R}^n \), \( t > 0 \),

\[
\text{osc}_p^k f(x, t) = \inf_{\deg P \leq k} \left( \frac{1}{|B(x, t)|} \int_{B(x, t)} |f(y) - P(y)|^p dy \right)^{1/p}, \ 1 \leq p < \infty,
\]

and

\[
\text{osc}_\infty^k f(x, t) = \inf_{\deg P \leq k} \sup_{|x-y| < t} |f(y) - P(y)|,
\]

where the infimum is taken over all polynomials \( P \) of degree less than or equal to a nonnegative integer \( k \) and \( |B(x, t)| \) means the volume of the ball \( B(x, t) \).

We write \( f \in T^s_{pq}(x) \) defined by (2) being replaced with

\[
\int_0^1 (t^{-s} \text{osc}_p^k f(x, t))^q \frac{dt}{t} < \infty \ \ (0 < q < \infty, \ s < k + 1),
\]
Proposition 2

easy routine of the function space theory, we can see a following proposition:

We denote \( \Omega \) has a unique solution which is given by the series

\[
\alpha_{pq}(f, x) = \sup \{ s \geq 0 : f \in T_{pq}^s(x) \},
\]

and

\[
\alpha_p(f, x) = \alpha_{p\infty}(f, x).
\]

By the definition it follows that \( \alpha(f, x) = \alpha_{\infty}(f, x) \) where \( \alpha(f, x) \) is given in (3). In the easy routine of the function space theory, we can see a following proposition:

**Proposition 2** (c.f. [7: p.3]).

(a) \( \alpha_{p\xi}(f, x) = \alpha_{pq}(f, x) \) for \( 0 < \xi, \eta \leq \infty \) and \( 1 \leq p \leq \infty \).

(b) \( \alpha(f) \leq \alpha(f, x) \leq \alpha_{p}(f, x) \leq \alpha_{q}(f, x) \leq \beta(f, x) \) for \( 1 \leq q \leq p < \infty \).

3 Self-similar functions.

**Definition.** A function \( F \) on \( \mathbb{R}^n \) is said to be self-similar relative to a function \( g \) on \( \mathbb{R}^n \) if

\[
F(x) = \sum_{j=1}^{d} \lambda_j F(S_j^{-1}x) + g(x), \quad x \in \mathbb{R}^n
\]

where \( \lambda_1, \lambda_2, \ldots, \lambda_d \) are real or complex numbers with \( 0 < |\lambda_j| < 1, j = 1, 2, \ldots, d \) and \( S_1, S_2, \ldots, S_d \) are contractive similarities with ratios \( \mu_1, \mu_2, \ldots, \mu_d \) satisfying \( 0 < \mu_j \leq 1, j = 1, 2, \ldots, d \). We remark that (6) implies \( \alpha(F) \leq \alpha(g) \).

From now on we will use notations \( S_J = S_{j_1}S_{j_2} \cdots S_{j_l} \), \( \lambda_J = \lambda_{j_1}\lambda_{j_2} \cdots \lambda_{j_l} \), \( \mu_J = \mu_{j_1}\mu_{j_2} \cdots \mu_{j_l} \), \(|J| = l \) for a multi-index \( J = (j_1, j_2, \ldots, j_l) \in \{1, 2, \ldots, d\}^l \), and \( S_J = \text{Identity} \), \( \lambda_J = 1, \mu_J = 1, |J| = 0 \) for \( J = 0 \). If a function \( F \) is self-similar relative to \( g \), then we have for any \( N \)

\[
F(x) = \sum_{|J|<N} \lambda_J g(S_J^{-1}x) + \sum_{|J|=N} \lambda_J F(S_J^{-1}x), \quad x \in \mathbb{R}^n.
\]

We say that the open set condition holds if there exits a bounded open set \( \Omega \) in \( \mathbb{R}^n \) such that

\[
S_i(\Omega) \subset \Omega, \quad i = 1, 2, \ldots, d
\]

and

\[
S_i(\Omega) \cap S_j(\Omega) = \emptyset \quad (i \neq j), \quad i, j = 1, 2, \ldots, d \quad (d \geq 2).
\]

We denote \( \Omega_0 = \Omega \), \( \Omega_i = S_i(\Omega) \) and \( \Omega_J = S_J(\Omega) \) for \( J = (j_1, j_2, \ldots, j_l) \). \( K = \bigcap_{l=0}^{\infty} \bigcup_{|J|=l} \Omega_J \) is called the invariant set with respect to similarities \( \{S_J\}_{j=1}^{d} \). Assume that the functions \( F \) and \( g \) are bounded and zero outside \( \Omega \). When the open set condition holds, then (6) has a unique solution which is given by the series

\[
F(x) = \sum_{J} \lambda_J g(S_J^{-1}x), \quad x \in \mathbb{R}^n.
\]
Then we can see easily that \( \alpha(F, x) \geq \alpha(g) \) for each \( x \notin K \).

Let
\[
a(x) = \liminf_{N \to \infty} \inf_{|J| \geq 0} \frac{\log |\lambda_J|}{\log \mu_J}, \quad x \in K
\]
where \( K_N(x) = \{ J : B(x, \mu_J) \cap \Omega_J \neq \emptyset, |J| = N \} \) and \( B(x, r) \) is a ball centered at \( x \) with a radius \( r \).

Lemma 2. Let \( \mu \) be any real number with \( 0 < \mu < 1 \) and \( x \in K \). We put
\[
K_N^\mu(x) = \{ J : B(x, \mu^N) \cap \Omega_J \neq \emptyset, \mu^{N+1} \leq \mu_J < \mu^N \}.
\]
Then we have
\[
a(x) = \liminf_{N \to \infty} \inf_{K_N^\mu(x) \ni J} \frac{\log |\lambda_J|}{\log \mu_J}.
\]

Proof. In easy routine we can prove the lemma. We will omit the details (c.f. [5]).

When \( x \in \cap_{i=0}^\infty \cup_{|J|=l} \Omega_J \), there exists a unique sequence \( J(x) = (j_1, j_2, \ldots) \) such that \( x \in \Omega_{j_i}(x) \) where \( j_i(x) = (j_1, j_2, \ldots, j_i) \) for \( i = 1, 2, \ldots \) and \( j_0(x) = 0 \). Let
\[
b(x) = \liminf_{N \to \infty} \frac{\log |\lambda_{J_N(x)}|}{\log \Delta_N(x)},
\]
for \( x \in \cap_{i=0}^\infty \cup_{|J|=l} \Omega_J \) where \( \Delta_N(x) = \text{dist}(x, \partial \Omega_{J_N(x)}) \) is the distance from \( x \) to the boundary of \( \Omega_{J_N(x)} \).

Theorem

Let \( F \) be a self-similar bounded function relative to a bounded function \( g \) on \( \mathbb{R}^n \), for which the open set condition holds. Assume that \( F(x) = g(x) = 0 \) for all \( x \notin \Omega \).

(a) Then we have
\[
\alpha(F, x) \geq \min(\alpha(g), a(x)), \quad x \in K
\]
where \( \alpha(F, x) \), \( \alpha(g) \) and \( a(x) \) are given in (3), (5) and (8) respectively.
(b) Suppose that \( g \in C^\infty(\Omega_i), i = 1, 2, \ldots, d \) (i.e. piecewise smooth) and \( \inf_{x \in \Omega} \beta(F, x) < \infty \). Then we have
\[
a(x_0) \geq \beta(F, x_0), \quad x_0 \in K
\]
where \( \beta(F, x_0) \) is given in (1).
(c) Suppose that \( x \in \cap_{i=0}^\infty \cup_{|J|=l} \Omega_J \) with \( \sup_i \frac{\Delta_N(x)}{\Delta_{N+1}(x)} < \infty \). Then we have
\[
\alpha(F, x) \geq \min_{i} (\alpha(g, \Omega_i), b(x))
\]
where \( \alpha(g, \Omega_i) \) and \( b(x) \) are given in (4) and (9) respectively.

Proof. (a) We fix any element \( x \in K \). We may assume \( \min(\alpha(g), a(x)) > 0 \). We choose a positive number \( s \) such that
\[
\min(\alpha(g), a(x)) > s > 0.
\]
We claim that \( F \in C^s(x) \). We may assume that there is a nonnegative integer \( k \) such that \( k + 1 > s > k \) and we can choose a positive number \( s' \) such that \( k + 1 > s' > s \) and \( g \in C^{s'}(\mathbb{R}^n) \).

From Lemma 2 there is a positive integer \( N_0 \) such that

\[
\forall N \geq N_0, \quad |\lambda_J| \leq \mu_J^s, \quad J \in K_N^M(x),
\]

where \( \mu \) is any fixed number with \( 0 < \mu < 1 \). Let \( T_x g(x - y) \) be the \( k \)-th Taylor polynomial of \( g \) at \( x \) and let \( PF(x - y) \) be a polynomial such that

\[
PF(x - y) = \sum_J \lambda_J T_{S_J^{-1}x} g(S_J^{-1}x - S_J^{-1}y).
\]

Let \( N \) be an integer such that \( N \geq N_0 \) and consider any \( y \in \mathbb{R}^n \) with \( \mu^{N+1} \leq |x-y| < \mu^N \). Then we have

\[
F(y) - PF(x - y) = \sum_{l=0}^{N-1} \sum_{J \in B_l} \lambda_J (g(S_J^{-1}y) - T_{S_J^{-1}x} g(S_J^{-1}x - S_J^{-1}y)) + \sum_{l=N}^{\infty} \sum_{J \in B_l} \lambda_J g(S_J^{-1}y)
\]

\[
- \sum_{l=N}^{\infty} \sum_{J \in B_l} \lambda_J T_{S_J^{-1}x} g(S_J^{-1}x - S_J^{-1}y) = I + II + III,
\]

where \( B_l = \{ J : \mu^{l+1} \leq \mu_J < \mu^l \} \).

The first sum is split in two parts:

\[
I = \sum_{l=0}^{N_0-1} + \sum_{l=N_0}^{N-1} = I_0 + I_1.
\]

From the fact that \( g \in C^{s'}(\mathbb{R}^n) \) the sum \( I_0 \) is estimated in

\[
|I_0| \leq C \sum_{l=0}^{N_0-1} \sum_{J \in B_l} |\lambda_J| |S_J^{-1}x - S_J^{-1}y|^s \leq C \sum_{l=0}^{N_0-1} \sum_{J \in B_l} |\lambda_J| |x - y|^s \leq C|x - y|^s \leq C|x - y|^a.
\]

We write \( B_l(y) = \{ J : \mu^{l+1} \leq \mu_J < \mu^l, y \in \Omega_J \} \). We can see easily that the cardinality of \( B_l(y) \) is bounded independently of \( y \) and \( l \). Then the sum \( I_1 \) is bounded by

\[
|I_1| \leq C \sum_{l=N_0}^{N-1} \sum_{J \in B_l(x) \cup B_l(y)} |\lambda_J| |S_J^{-1}x - S_J^{-1}y|^s \leq C \sum_{l=N_0}^{N-1} \sum_{J \in B_l(x) \cup B_l(y)} |\lambda_J| |x - y|^s \leq C \sum_{l=N_0}^{N-1} \sum_{J \in B_l(x) \cup B_l(y)} \mu^{-(l+1)(s'-s)} |x - y|^s' \leq C \mu^{-N(s'-s)} |x - y|^s' \leq C|x - y|^a.
\]

We estimate the sum \( II \) by

\[
|II| \leq \sum_{l=N}^{\infty} \sum_{J \in B_l(y)} |\lambda_J| |g(S_J^{-1}y)| \leq C \sum_{l=N}^{\infty} |\lambda_J| \leq C \sum_{J \in B_N(y)} |\lambda_J| \leq C \mu^s \leq C \mu^N \leq C|x - y|^a.
\]
The sum \( III \) is bounded in
\[
|III| \leq C \sum_{l=N}^{\infty} \sum_{J \in B_l(x)} \sum_{|\alpha| \leq k} \mu_j^{-|\alpha|}|x - y|^{|\alpha|}y_j
\]
\[
\leq C \sum_{|\alpha| \leq k} \sum_{l=N}^{\infty} \sum_{J \in B_l(x)} \mu_j^{-|\alpha|}|x - y|^{|\alpha|}
\]
\[
\leq C \sum_{|\alpha| \leq k} \mu_j^{-N(s-|\alpha|)}|x - y|^{|\alpha|} \leq C \mu_j^{-N} \leq C|x - y|^s.
\]
Hence we obtain
\[
|F(y) - PF(x - y)| \leq |I_0| + |I_1| + |II| + |III| \leq C|x - y|^s.
\]
This completes the part (a) of the theorem.

(b) We start from \( F \in \Gamma^s(x_0) \) and we will claim that \( s \leq a(x_0) \). Let us \( k \) a large enough integer that will be determined later with \( k > s \) and let us consider a function \( \varphi \) satisfying the conditions in Lemma 1 and \( \int_0^\infty |\dot{\varphi}(t\xi)|^2 dt / t \neq 0 \) for \( \xi \neq 0 \).

Then we have from Lemma 1 for \( C_0 = 1 + \text{diam } \Omega \),
\[
|t^{-n} \int \varphi(R(y - x) / t) F(y) dy| \leq C t^s
\]
for any orthogonal \( R \) whenever \( 0 < t \leq \delta_0 \) and \( |x - x_0| < C_0 t \). We may assume that \( 0 \in \Omega \) and \( F \notin \Gamma^k(0) \). Hence \( v_j = S_j 0 \in \Omega_j \) for any \( J \). We can choose a large integer \( N \) such that \( 0 < \mu_j \leq \delta_0 \) for any \( J = N \). For this integer \( N \), we fix any element \( J^0 = (j^0_1, j^0_2, \ldots, j^0_N) \in K_N(x_0) \). Since \( |x_0 - v_j| < C_0 \mu_j \) for all \( J \in K_N(x_0) \) and we can write \( S_{j^0}^{-1} y = \frac{R_{j^0}^{-1}(y - v_j)}{\mu_{j^0}} \) for some orthogonal \( R_{j^0} \), it holds that
\[
|\mu_{j^0}^{-n} \int \varphi(S_{j^0}^{-1} y) F(y) dy| \leq C \mu_{j^0}^s. \quad (10)
\]

We may assume that \( \text{supp } \varphi \subset \Omega \). Hence we have from (7),
\[
\mu_{j^0}^{-n} \int \varphi(S_{j^0}^{-1} y) F(y) dy = \int \varphi(y) F(S_{j^0} y) dy
\]
\[
= \sum_{|J| < N} \lambda_J \int \varphi(y) g(S_j^{-1} S_{j^0} y) dy + \sum_{|J| = N} \lambda_J \int \varphi(y) F(S_j^{-1} S_{j^0} y) dy
\]
\[
= \sum_{i=0}^{N-1} \lambda_{j^0_i} \int \varphi(y) g(S_{j^0_i}^{-1} S_{j^0} y) dy + \lambda_{j^0} \int \varphi(y) F(y) dy \quad (11)
\]
where \( J^0_0 = 0 \) and \( J^0_i = (j^0_1, \ldots, j^0_i), \quad 1 \leq i < N \). We have from \( g \in C^\infty(\Omega_i) \) for \( i = 1, 2, \ldots, d \),
\[
|\int \varphi(y) g(S_{j^0_i}^{-1} S_{j^0} y) dy| = \left| \int \varphi(y) ((g(S_{j^0_i}^{-1} S_{j^0} y) - T g(S_{j^0_i}^{-1} S_{j^0} y - S_{j^0_i}^{-1} S_{j^0} 0)) dy| \right|
\]
\[
\leq C \int |\varphi(y)||S_{j^0_i}^{-1} S_{j^0} y - S_{j^0_i}^{-1} S_{j^0} 0|^k dy \leq C (\mu_{j^0_{i+1}} \ldots \mu_{j^0_N}^k) \int |\varphi(y)||y|^k dy,
\]
for $0 \leq l < N$ where $Tg$ is the $(k-1)$-th Taylor polynomial of $g$ at $S_{j_l}^{-1}S_{j_0}0$. From above, we have

\[
| \sum_{l=0}^{N-1} \lambda_{j_l} \int \varphi(y)g(S_{j_l}^{-1}S_{j_0}y)dy | \leq C \int |\varphi(y)||y|^k dy \sum_{l=0}^{N-1} |\lambda_{j_l}|(\mu_{j_{l+1}} \ldots \mu_{j_N})^k
\]

\[
\leq C \int |\varphi(y)||y|^k dy \frac{\lambda_{j_l}|(\max_i \mu_i)^N\lambda_k}{(\min_i |\lambda_i|)^N} \sum_{l=0}^{N-1} \frac{(\min_i |\lambda_i|)^l}{(\max_i \mu_i)k} \leq C|\lambda_{j_0}| \int |\varphi(y)||y|^k dy
\]

for an integer $k$ such that $\min_i |\lambda_i| > \max_i \mu_i^k$.

From [7: Theorem 3.5] we may assume that $C \int |\varphi(y)||y|^k dy \leq \frac{1}{2}$ in the right hand side of the above and $|\int \varphi(y)F(y)dy| = 1$ because of that $F \notin \Gamma^k(0)$. Thus we obtain from (11)

\[
|\mu_{j_0}^{-1} \int \varphi(S_{j_0}^{-1}y)F(y)dy|
\]

\[
\geq |\lambda_{j_0}| |\int \varphi(y)F(y)dy| - \sum_{l=0}^{N-1} \lambda_{j_l}^{-1} \int \varphi(y)g(S_{j_l}^{-1}S_{j_0}y)dy |
\]

\[
\geq |\lambda_{j_0}| - \frac{1}{2}|\lambda_{j_0}| \geq \frac{1}{2}|\lambda_{j_0}|.
\]

(10 ) and (12) imply that $a(x_0) \geq s$. This yields the part (b) of the theorem.

The proof of the part (c) of the theorem is the same as the proof of the part (a). We use the same notations in the proof of the part (a). We fix any element $x$ in $\cap_{l=0}^{\infty} \cup_{j=1}^{\infty} \Omega_j$.

Let $J(x) = (j_1, j_2, \ldots)$ be a sequence such that $x \in \Omega_{j_l(x)}$ for all $l \geq 0$ where $J_l(x) = (j_1, j_2, \ldots, j_l)$ and $J_0(x) = 0$. We may choose a positive number $s$ such that

\[
\min_i (\alpha(g, \Omega_i), b(x)) > s > 0.
\]

We will claim that $F \in C^s(x)$. We may assume that $k + 1 > s > k$ with a nonnegative integer $k$ and we choose a positive number $s'$ such that $k + 1 > s' > s$ and $g \in C^{s'}(\Omega_i), i = 1, 2, \ldots, d$. From the definition of $b(x)$ in (9), there is a positive integer $N_0$ such that

\[
|\lambda_{j_N(x)}| \leq \Delta_N(x)^s, \forall N \geq N_0.
\]

We consider any $y \in \mathbb{R}^n$ such that $\Delta_{N+1}(x) \leq |x - y| < \Delta_N(x)$. Hence $y \in \Omega_{j_N(x)}$. We put

\[
F(y) - P F(x - y) = \sum_{|J| < N} \lambda_{j_l}(g(S_{j_l}^{-1}y) - T_{S_{j_l}^{-1}x}g(S_{j_l}^{-1}x - S_{j_l}^{-1}y))
\]

\[
+ \sum_{|J| = N} \lambda_{j_l}F(S_{j_l}^{-1}y) - \sum_{|J| \geq N} \lambda_{j_l}T_{S_{j_l}^{-1}x}g(S_{j_l}^{-1}x - S_{j_l}^{-1}y)
\]

\[
= I + II + III.
\]

We split the sum $I$ to two parts:

\[
I = \sum_{|J| < N_0} + \sum_{N_0 \leq |J| < N} = I_0 + I_1.
\]
Since $\Delta_l(x) \leq (\text{diam } \Omega) \mu_{J_l(x)}$ for $l > 0$, these sums are bounded in

$$|I_0| \leq C \sum_{l=0}^{N_0-1} |\lambda_{J_l(x)}| \mu_{J_l(x)}^{-s'} |x - y|^{s'} \leq C |x - y|^{s'} \leq C |x - y|^s.$$ 

$$|I_1| \leq C \sum_{l=N_0}^{N-1} |\lambda_{J_l(x)}| \mu_{J_l(x)}^{-s'} |x - y|^{s'} \leq C \sum_{l=N_0}^{N-1} \Delta_l(x)^s \mu_{J_l(x)}^{-s'} |x - y|^{s'} \leq C \Delta_N(x)^{-(s'-s)} |x - y|^{s'} \leq C |x - y|^s.$$ 

$$|I_2| \leq |\lambda_{J_N(x)}||F(S_{J_N(x)}^{-1}y)| \leq C \Delta_N(x)^s \leq C \Delta_{N+1}(x)^s \leq C |x - y|^s.$$ 

$$|I_{III}| \leq C \sum_{l=N}^{\infty} |\lambda_{J_l(x)}| \sum_{|\alpha| \leq k} \mu_{J_l(x)}^{-|\alpha|} |x - y|^{\alpha} \leq C \sum_{|\alpha| \leq k} \sum_{l=N}^{\infty} |\lambda_{J_l(x)}| \Delta_l(x)^{-|\alpha|} |x - y|^{\alpha} \leq C \sum_{|\alpha| \leq k} \Delta_N(x)^{s-|\alpha|} |x - y|^{\alpha} \leq C |x - y|^s.$$ 

These estimations yield the proof of the part (c) of the theorem.

4 Examples.

We consider the similarities $S_j x = \frac{x + j - 1}{2}$, $j = 1, 2$ on $\mathbb{R}$. Then the open set condition holds for the open interval $\Omega = (0, 1)$. In the case when $x$ is a nondyadic point in $\Omega = (0, 1)$, (i.e. $x \in \cap_{l=0}^{\infty} \cup_{|J| = l} \Omega_J$), we have that

$$a(x) = \liminf_{N \to \infty} \frac{\log |\lambda_{J_N(x)}|}{\log 2^{-N}}$$

where $J_N(x)$ is given in the part (c) of the proof in the theorem, and for a nondyadic point $x$ in $\Omega = (0, 1)$ with $\sup_N \frac{\Delta_N(x)}{\Delta_{N+1}(x)} < \infty$, we have $a(x) = b(x)$.

Let $g$ be a bounded function on $\mathbb{R}$ such that $g \in C^\infty(\Omega_j), j = 1, 2$ and $g = 0$ outside $\Omega$. Consider a self-similar function $F$ given by

$$F(x) = \sum_{j=1}^{2} \lambda_j F(S_j^{-1}x) + g(x), \ x \in \mathbb{R} \tag{13}$$

with $0 < |\lambda_j| < 1$, $j = 1, 2$ and $F(x) = 0$ outside $\Omega = (0, 1)$. From the theorem, if $\inf_{x \in \Omega} \beta(F, x) < \infty$ we have

$$a(x) \geq \beta(F, x) \geq \alpha(F, x) \geq \min(\alpha(g), a(x)), \ x \in K = [0, 1] \tag{14}$$
and for a non-dyadic point \( x \) in \( \Omega = (0, 1) \) with sup\( N \frac{\Delta_N(x)}{\Delta_{N+1}(x)} < \infty \),

\[
\alpha(F, x) = \beta(F, x) = a(x) = b(x). \tag{15}
\]

(a) Let \( \mu \) be the Bernoulli measure which is a probability measure supported on \([0, 1]\) such that \( \mu(I_1) = \lambda_1 \mu(I) \) and \( \mu(I_2) = \lambda_2 \mu(I) \) when \( I \) is a dyadic interval, \( I_1 \) is the left half of \( I \) and \( I_2 \) is the right half of \( I \) with \( \lambda_1 > 0, \lambda_2 > 0, \lambda_1 \neq \lambda_2 \) and \( \lambda_1 + \lambda_2 = 1 \). Let \( F_0(x) = \mu[0, x) \) be the continuous function whose distributional derivative is \( \mu \) with \( F_0(x) = 0 \) \((x \leq 0)\) and \( F_0(x) = 1 \) \((x \geq 1)\). Then from (14) we can see that

\[
\beta(F_0, x) = \alpha(F_0, x) = a(x), \quad x \in K = [0, 1].
\]

(See [7: Proposition 1.9 and Proposition 1.10]). Let \( F(x) = F_0(x) - x \) \((0 < x < 1)\), and \( F(x) = 0 \) \((\text{otherwise})\). Then \( F \) is a self-similar function such that

\[
F(x) = \lambda_1 F(S_1^{-1} x) + \lambda_2 F(S_2^{-1} x) + g(x), \quad \forall x \in \mathbb{R}
\]

where \( g(x) = (2\lambda_1 - 1) x \) \((0 < x \leq \frac{1}{2})\), \( g(x) = (2\lambda_1 - 1)(1 - x) \) \((\frac{1}{2} \leq x < 1)\), \( g(x) = 0 \) \((\text{otherwise})\). Then from (14) we can see that

\[
\alpha(F, x) = \beta(F, x), \quad x \in K = [0, 1].
\]

In particular we have \( a(x) = \alpha(F, x) = \beta(F, x) \) for each \( x \in \Omega = (0, 1) \).

(b) We consider the Takagi function such that

\[
F(x) = \sum_{j=1}^{\infty} \lambda^{|j|} g(S_j^{-1} x), \quad \forall x \in \mathbb{R} \tag{16}
\]

where \( 0 < \lambda < 1 \) and \( g \) is a bounded function such that \( g(x) = x \) \((0 < x \leq \frac{1}{2})\), \( g(x) = 1 - x \) \((\frac{1}{2} \leq x < 1)\), \( g(x) = 0 \) \((\text{otherwise})\). Then \( F \) is a self-similar function such that

\[
F(x) = \sum_{j=1}^{2} \lambda F(S_j^{-1} x) + g(x), \quad \forall x \in \mathbb{R}.
\]

Let \( a = \frac{\log \lambda}{\log 2^{-1}} \). Then from (14), if \( a \leq 1 \), \( a = \alpha(F, x) = \beta(F, x) \) for each \( x \in K \).

(c) We consider the Weierstrass function \( F(x) = \sum_{j=0}^{\infty} \lambda^j g(2^j x) \) with \( 0 < \lambda < 1 \) and \( g(x) = \sin 2\pi x \) \((x \in \mathbb{R})\). Then \( F \) is a self-similar function such that

\[
F(x) = \sum_{j=1}^{2} (2^{-1}) \lambda F(S_j^{-1} x) + g(x), \quad \forall x \in \mathbb{R}.
\]

The proof of the theorem can be also applied to this self-similar function case. Then we have

\[
a = \alpha(F, x) = \beta(F, x), \quad \forall x \in \mathbb{R}.
\]

where the constant \( a = \frac{\log \lambda}{\log 2^{-1}} \) is given in the part (b) above.
(d) We consider Lévy’s function $F(x)$ in (16) being replaced by $g(x) = x - \frac{1}{2}$ ($0 < x < 1$), $g(x) = 0$ (otherwise) and $\lambda = 2^{-1}$. In [3: Proposition 4] it follows that $\alpha(F, x) = b(x)$ for each nondyadic point $x$ in $\Omega = (0, 1)$. Moreover we can see by (15) that $1 = b(x) = \alpha(F, x) = \beta(F, x)$ for a nondyadic point $x$ in $\Omega = (0, 1)$ with $\sup_N \frac{\Delta_N(x)}{\Delta_{N+1}(x)} < \infty$.

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